Chapter 4

Supersymmetric Quantum Mechanics on the 2-sphere

Appendix 4(i)

Complex Manifolds [12], [13]

A complex manifold M is a differentiable manifold which can be covered by a set of complex coordinate patches, and where the coordinate transformations on the overlap between the patches are holomorphic. On a complex manifold the cotangent space decomposes into the sum of a holomorphic and an anti-holomorphic cotangent space, so p-forms may be treated as (q, r)-forms, which q and r are the holomorphic and anti-holomorphic degrees respectively, and p = q + r. The exterior derivative on a complex manifold splits naturally into

$$d = \partial + \bar{\partial}$$

where ∂, the holomorphic derivative map (q, r)-forms to (q+1, r)-forms and $\bar{\partial}$, the anti-holomorphic derivative maps (q, r)-forms to (q, r+1)-forms. The nilpotence of the exterior derivative implies the relations

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0 \,.$$

The Hodge Star can now be used to define the adjoint operators

$$\partial^* = -{}^*\partial^* \quad , \quad \bar{\partial}^* = -{}^*\bar{\partial}^*$$

so that in general on a complex manifold there are three different Laplacians:

$$\triangle_d = d\delta + \delta d \,, \quad \triangle_\partial = \partial\partial^* + \partial^*\partial \,, \quad \triangle_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \,.$$

On a complex manifold a Hermitean metric may be defined

$$ds^2 = \sum_{i,j} h_{ij}(z) dz_i d\bar{z}_j$$

where $h_{ij}(z)$ is a positive definite Hermitean matrix for each z.

The associated (1, 1)-form

$$\omega = \tfrac{i}{2} \sum_{i,j} h_{ij}(z) dz_i \wedge d\bar{z}_j$$

is called the Kähler form. A manifold with closed Kähler form, $d\omega = 0$, is known as a Kähler manifold. An important property of Kähler manifolds is that the three different Laplacians defined on them are equal up to a constant.

$$\frac{1}{2}\triangle_d = \triangle_\partial = \triangle_{\bar\partial} \, .$$

This relation leads to Hodge duality

$$h^{p,q}(M) = h^{q,p}(M)$$

where $h^{p,q}(M)$ are the Hodge Numbers of M, the dimension of harmonic (p, q)-forms. In terms of supersymmetry the fact that the different Laplacians are equal means that the two supersymmetry operators can be split into four

$$Q_1 = \partial + \partial^* \, , \qquad Q_2 = \bar\partial + \bar\partial^* \, , \qquad Q_3 = i(\partial - \partial^*) \, , \qquad Q_4 = i(\bar\partial - \bar\partial^*)$$

which satisfy the supersymmetry algebra

$$H = Q_1^2 = Q_2^2 = Q_3^2 = Q_4^2 = \frac{1}{2}\triangle_d \quad ; \qquad \{Q_i, Q_j\} = 0 \ , \qquad i \neq j \quad ,$$

so that the non-zero energy solutions form quadruplets under supersymmetry rather than doublets, while the zero energy solutions are still singlets. Poincare duality doubles the degeneracy of excited states to eightfold. This does not, however, affect the analysis of section 3.1 where a Killing Vector was introduced into the Hamiltonian, as a Killing Vector is necessarily real so the degeneracy due to supersymmetry is the same whether or not the manifold is Kähler.

On a compact complex manifold of n complex dimensions the volume form V is proportional to the n-th power exterior product of the Kähler form

$$V = \frac{1}{n!}\omega^n \qquad .$$

If the manifold is Kähler this relationship along with the Kähler condition, $d\omega = 0$, implies that ω^k is a representative of the cohomology of the manifold for all k, $0 \le k \le n$. The Kähler condition implies ω^k is closed, but if it was exact, $\omega^k = d\alpha$, by Stokes Theorem, the volume would have to be zero

$$vol(M) = \frac{1}{n!}\int_M \omega^n = \frac{1}{n!}\int_M (d\alpha)\wedge\omega^{n-k} = \frac{1}{n!}\int_M d(\alpha\wedge\omega^{n-k}) = 0 \qquad .$$

In virtually all of the following, complex coordinates will be used, as this results in considerable simplifications. The functions which are defined will usually be written with arguments which are holomorphic, this is an abbreviation, i.e. $\Psi(z) = \Psi(z, \bar z)$. None of the functions are actually holomorphic.

4. Supersymmetric Quantum Mechanics on the 2-sphere

The simplest non-trivial illustrations of Witten's ideas about fixed point theorems and supersymmetric quantum mechanics is on the 2-sphere. In this section the supersymmetric Schrödinger Equation on the 2-sphere is solved; the solutions form multiplets under the isometry group SU(2). An infinitesimal generator of an isometry, a Killing Vector, is then introduced into the supersymmetry algebra as in section 3. This breaks the symmetry under the isometry group and leads to a new Hamiltonian which cannot be solved exactly for excited states, though zero energy solutions can be found. The zero energy solutions are related to the topology of the 2-sphere in the way expect from Witten's paper [1]. Perturbation theory corrections to the spectrum of the excited states are calculated. The asymptotic regime is also examined and related to Witten's analysis. Finally, the various approximations to the energy level are compared to those found numerically by a computer program.

4.1 The 2-sphere

As a complex manifold the 2-sphere is equivalent to $\mathbb{CP}^1$. It can therefore be defined in terms of two homogeneous complex coordinates Z_0 and Z_1 with points identified which are equal up to a scaling by a complex number. The 2-sphere can be covered with two patches by using the inhomogeneous coordinates

$$z = \frac{Z_1}{Z_0} \qquad \text{everywhere except} \qquad Z_0 = 0$$

$$w = \frac{Z_0}{Z_1} \qquad \text{everywhere except} \qquad Z_1 = 0 \qquad .$$

A metric may be put on the 2-sphere by stereographic projection. In terms of the coordinates z, this is

$$ds^2 = \frac{dz d\bar{z}}{(1+z\bar{z})^2} \qquad .$$

The group of metric preserving rotations in two flat complex dimensions is U(2). The infinitesimal generators of this group may be spanned by the operators

$$Z_1\frac{d}{dZ_1} - \bar{Z}_1\frac{d}{d\bar{Z}_1}\,, \qquad Z_1\frac{d}{dZ_0} - \bar{Z}_0\frac{d}{d\bar{Z}_1}\,, \qquad Z_0\frac{d}{dZ_1} - \bar{Z}_1\frac{d}{d\bar{Z}_0}\,, \qquad Z_0\frac{d}{dZ_0} - \bar{Z}_0\frac{d}{d\bar{Z}_0}$$

The isometry group of the 2-sphere is SU(2). The generators of this group may be obtained from the generators of U(2) by using the chain rule to rewrite them in terms of the inhomogeneous coordinate z.

$$Z_1\frac{d}{dZ_1} - \bar{Z}_1\frac{d}{d\bar{Z}_1} = z\frac{d}{dz} - \bar{z}\frac{d}{d\bar{z}} = I_3$$

$$Z_1\frac{d}{dZ_0} - \bar{Z}_0\frac{d}{d\bar{Z}_1} = -z^2\frac{d}{dz} - \frac{d}{d\bar{z}} = \sqrt{2}I_+$$

$$Z_0\frac{d}{dZ_1} - \bar{Z}_1\frac{d}{d\bar{Z}_0} = \frac{d}{dz} + \bar{z}^2\frac{d}{d\bar{z}} = \sqrt{2}I_-$$

These three operators, I_3 and the two ladder operators $I_\pm$, form a basis for SU(2). The operator $Z_0\frac{d}{dZ_0} - \bar{Z}_0\frac{d}{d\bar{Z}_0}$ is equal to $-I_3$ and so doesn't give anything new. The freedom to scale the homogeneous coordinates on the sphere by a complex number has removed a U(1) factor from the isometry group. The commutation relations of the SU(2) generators are

$$[I_+, I_-] = I_3 \quad , \qquad [I_3, I_-] = -I_- \qquad , \qquad [I_3, I_+] = I_+$$

Rewriting in terms of I_1 and I_2 gives

$$I_1 = \frac{1}{\sqrt{2}}(-I_+ + I_-) = \frac{1}{2}[\frac{d}{dz} + \frac{d}{d\bar{z}} + z^2\frac{d}{dz} + \bar{z}^2\frac{d}{d\bar{z}}]$$

$$I_2 = \frac{-i}{\sqrt{2}}(I_+ + I_-) = \frac{i}{2}[\frac{d}{dz} - \frac{d}{d\bar{z}} + z^2\frac{d}{dz} - \bar{z}^2\frac{d}{d\bar{z}}]$$

and the commutation relations

$$[I_i, I_j] = i\epsilon_{ijk}I_k \qquad i, j, k \text{ run over } 1, 2, 3.$$

4.1.1 The Ordinary Laplacian

As on the 2-torus the 4x4 matrix notation for operators will be used. Wavefunctions on the sphere will be of the form

$$\Psi(z) = A(z) + B(z)dz + C(z)d\bar{z} + D(z)dz \wedge d\bar{z}$$

and will be denoted by the 4x1 column matrix $\begin{bmatrix} A(z) \\ B(z) \\ C(z) \\ D(z) \end{bmatrix}$.

Using this notation the holomorphic and anti-holomorphic derivatives may be written as

$$\partial = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{d}{dz} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{d}{dz} & 0 \end{pmatrix}$$

$$\bar{\partial} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{d}{d\bar{z}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{d}{d\bar{z}} & 0 \end{pmatrix}$$

and by applying the Hodge Star Map their adjoints may be found:

$$\partial^* = \begin{pmatrix} 0 & -(1+z\bar{z})^2\frac{d}{d\bar{z}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{d\bar{z}}(1+z\bar{z})^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\bar{\partial}^* = \begin{pmatrix} 0 & 0 & -(1+z\bar{z})^2\frac{d}{dz} & 0 \\ 0 & 0 & 0 & \frac{d}{dz}(1+z\bar{z})^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\frac{d}{dz}(1+z\bar{z})^2$ means that $\frac{d}{dz}$ acts on the product of $(1+z\bar{z})^2$ and the wavefunction.

Because the 2-sphere is a Kähler manifold there are four supersymmetry operators as opposed to two in the generic case. These supersymmetry operators are

$$Q_1 = \partial + \partial^* = \begin{pmatrix} 0 & -(1+z\bar{z})^2\frac{d}{d\bar{z}} & 0 & 0 \\ \frac{d}{dz} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{d\bar{z}}(1+z\bar{z})^2 \\ 0 & 0 & \frac{d}{dz} & 0 \end{pmatrix}$$

$$Q_2 = \bar\partial + \bar\partial^* = \begin{pmatrix} 0 & 0 & -(1+z\bar z)^2\frac{d}{dz} & 0 \\ 0 & 0 & 0 & \frac{d}{dz}(1+z\bar z)^2 \\ \frac{d}{d\bar z} & 0 & 0 & 0 \\ 0 & -\frac{d}{d\bar z} & 0 & 0 \end{pmatrix}$$

and $Q_3 = i(\partial - \partial^*)\,,\, Q_4 = i(\bar\partial - \bar\partial^*)\,.$

The Hamiltonian is the square of these operators, the Laplacian.

$$H = Q_1^2 = Q_2^2 = Q_3^2 = Q_4^2 =$$
$$-(1+z\bar z)^2 \begin{pmatrix} \frac{d^2}{dzd\bar z} & 0 & 0 & 0 \\ 0 & \frac{d^2}{dzd\bar z} & 0 & 0 \\ 0 & 0 & \frac{d^2}{dzd\bar z} & 0 \\ 0 & 0 & 0 & \frac{d^2}{dzd\bar z} \end{pmatrix} \qquad -2\bar z(1+z\bar z) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{d}{d\bar z} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{d}{d\bar z} \end{pmatrix}$$

$$-2z(1+z\bar z) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{d}{dz} & 0 \\ 0 & 0 & 0 & \frac{d}{dz} \end{pmatrix} \quad -2(1+2z\bar z) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad .$$

The zero energy eigensolutions correspond to the cohomology of the 2-sphere; namely the constant zero-form $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and the volume form $\begin{pmatrix} 0 \\ 0 \\ 0 \\ (1+z\bar z)^{-2} \end{pmatrix}$. The excited states may be found by considering the Hamiltonian on zero-forms and then using the supersymmetry operators to find the one-form and the two-form solutions.

On zero-forms the Hamiltonian is simply

$$H_0 = -(1+z\bar z)^2 \frac{d^2}{dzd\bar z} \qquad .$$

The isometries of the 2-sphere are rotations of the manifold which preserve the metric, thus the generators of SU(2) commute with the Laplacian and so its eigensolutions form representations os SU(2). The highest weight solutions are of the form

$$Y_n^n = \frac{z^n}{(1+z\bar{z})^n}$$

which can easily be shown to satisfy the Schrödinger Equation with energy $E = n(n+1)$.

$$\begin{aligned}
-(1+z\bar{z})^2 \frac{d^2}{dz d\bar{z}}[\frac{z^n}{(1+z\bar{z})^n}] &= -(1+z\bar{z})^2 \frac{d}{dz}[\frac{-nz^{n+1}}{(1+z\bar{z})^{n+1}}] \\
&= -(1+z\bar{z})^2[-\frac{n(n+1)z^n}{(1+z\bar{z})^{n+1}} + \frac{n(n+1)z^{n+1}\bar{z}}{(1+z\bar{z})^{n+2}}] \\
&= n(n+1)(1+z\bar{z})^2[\frac{z^n(1+z\bar{z})}{(1+z\bar{z})^{n+2}} - \frac{z^n z\bar{z}}{(1+z\bar{z})^{n+2}}] \\
&= n(n+1)\frac{z^n}{(1+z\bar{z})^n} \qquad .
\end{aligned}$$

These solutions have highest weight because they are annihilated by the raising operator I_+

$$\begin{aligned}
I_+ Y_n^n(z) = (z^2\frac{d}{dz} + \frac{d}{d\bar{z}})[\frac{z^n}{(1+z\bar{z})^n}] &= z^2[\frac{nz^{n-1}}{(1+z\bar{z})^n} - \frac{nz^n\bar{z}}{(1+z\bar{z})^{n+1}}] + [-\frac{nz^{n+1}}{(1+z\bar{z})^{n+1}}] \\
&= \frac{nz^{n+1}(1+z\bar{z}) - nz^{n+2}\bar{z} - nz^{n+1}}{(1+z\bar{z})^{n+1}} \\
&= 0
\end{aligned}$$

and the eigenvalue of the operator $I_3 = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}$ is n. The rest of the solutions in the multiplets can be found by applying the lowering operator I_- , which lowers the eigenvalue of I_3 by one at each step. These multiplets have energy $E = n(n+1)$ and each member is labelled by the eigenvalue of I_3 , the angular momentum around the equator of the sphere, which takes the values $-n \leq m \leq n$. These solutions $Y_n^m(z)$ equal the Associated Legendre Functions $P_n^m(\theta, \phi)$, where $z = tan\frac{\theta}{2}e^{i\phi}$. For example, the first excited states form the adjoint representation of SU(2)

$$\begin{array}{ccc}
Y_1^{-1}(z) & Y_1^0(z) & Y_1^1(z) \\
\frac{\bar{z}}{1+z\bar{z}} & \frac{1-z\bar{z}}{1+z\bar{z}} & \frac{z}{1+z\bar{z}}
\end{array}$$

and have energy E = 2 .

Having solved the Schrödinger Equation for (0,0)-forms, we are now in the position of being able to find the (1,0)-form, (0,1)-form and (1,1)-form solutions. Acting with the supersymmetry operator Q_1 on a (0,0)-form solution gives a (1,0)-form solution

$$Q_1 \begin{pmatrix} Y_n^m(z) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{d}{dz} Y_n^m(z) \\ 0 \\ 0 \end{pmatrix}$$

and acting with Q_2 gives a (0,1)-form solution

$$Q_2 \begin{pmatrix} Y_n^m(z) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{d}{d\bar{z}} Y_n^m(z) \\ 0 \end{pmatrix}$$

while acting consecutively with both supersymmetry operators will give a (1,1)-form solution.

$$Q_2 Q_1 \begin{pmatrix} Y_n^m(z) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{d^2}{dz d\bar{z}} Y_n^m(z) \end{pmatrix} = n(n+1) \begin{pmatrix} 0 \\ 0 \\ 0 \\ (1+z\bar{z})^{-2} Y_n^m(z) \end{pmatrix}$$

so supersymmetry produces a four-fold degeneracy on top of the (2n+1)-fold degeneracy due to SU(2) symmetry.

4.2 Inclusion of a Killing Vector into the Supersymmetry Algebra

Following Witten we will now generalize the exterior derivative by the addition of the operation of taking the interior product with a Killing Vector **k**

$$d_s = d + s i_{\mathbf{k}}$$

where s is a positive parameter. This leads to a supersymmetry algebra of the form of equations (1) of section 3.4. On the 2-sphere the simplest Killing Vector to use is the generator of rotations around the equator iI_3 . The factor of i is necessary because I_3 is imaginary and a Killing Vector must be real.

Writing out the operators d_s and δ_s explicitly using the Killing Vector $k = iI_3$ gives

$$d_s = \begin{pmatrix} 0 & isz & -is\bar{z} & 0 \\ \dfrac{d}{dz} & 0 & 0 & is\bar{z} \\ \dfrac{d}{d\bar{z}} & 0 & 0 & isz \\ 0 & -\dfrac{d}{d\bar{z}} & \dfrac{d}{dz} & 0 \end{pmatrix}$$

$$\delta_s = \begin{pmatrix} 0 & -(1+z\bar{z})^2\dfrac{d}{d\bar{z}} & -(1+z\bar{z})^2\dfrac{d}{dz} & 0 \\ \dfrac{-is\bar{z}}{(1+z\bar{z})^2} & 0 & 0 & \dfrac{d}{dz}(1+z\bar{z})^2 \\ \dfrac{isz}{(1+z\bar{z})^2} & 0 & 0 & -\dfrac{d}{d\bar{z}}(1+z\bar{z})^2 \\ 0 & \dfrac{-isz}{(1+z\bar{z})^2} & \dfrac{-is\bar{z}}{(1+z\bar{z})^2} & 0 \end{pmatrix} .$$

Defining the supersymmetry operators as $Q_{s1} = i^{-\frac{1}{2}} d_s + i^{\frac{1}{2}} \delta_s$, leads to the Hamiltonian

$$H_s = H + \frac{s^2 z\bar{z}}{(1+z\bar{z})^2}\mathbb{I} + is \begin{pmatrix} 0 & 0 & 0 & -(1-z\bar{z})(1+z\bar{z}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{(1-z\bar{z})}{(1+z\bar{z})^3} & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

where H is the ordinary Laplacian and $\mathbb{I}$ is the 4x4 unit matrix. The Hamiltonian operator has been defined such that $H_s = \frac{1}{2}(d_s\delta_s + \delta_s d_s)$. This is because the holomorphic Laplacian $\triangle_\partial$, on a Kähler manifold, equals half the ordinary Laplacian $\triangle_d$, see Appendix 4(i). The new s-dependent terms correspond to the terms in section 3 multiplied by a half

$$\tfrac{1}{2}(s^2K^2 + s(e_{d\tilde{\mathbf{k}}} + {}^*e^*_{d\tilde{\mathbf{k}}}))$$

where $\mathbf{k} = i(z\frac{d}{dz} - \bar{z}\frac{d}{d\bar{z}})$ and $\tilde{\mathbf{k}}$ is its dual, $\tilde{\mathbf{k}} = -i(1+z\bar{z})^{-2}(\bar{z}dz - zd\bar{z})$.

The central charge P takes the form

$$P = \tfrac{1}{2}(d_s^2 - \delta_s^2) = s\begin{pmatrix} z\frac{d}{dz} - \bar{z}\frac{d}{d\bar{z}} & 0 & 0 & 0 \\ 0 & z\frac{d}{dz} - \bar{z}\frac{d}{d\bar{z}} + i & 0 & 0 \\ 0 & 0 & z\frac{d}{dz} - \bar{z}\frac{d}{d\bar{z}} - i & 0 \\ 0 & 0 & 0 & z\frac{d}{dz} - \bar{z}\frac{d}{d\bar{z}} \end{pmatrix}$$

$$= is\mathcal{L}_{\mathbf{k}}$$

The Schrödinger Equation can be diagonalized on even-forms, i.e. (0,0)-forms and (1,1)-forms by writing the solutions in terms of the self and anti-self-dual combinations of the unit (0,0)-form and the Kähler form

$$\phi_{\pm}(z) = \begin{pmatrix} A_{\pm}(z) \\ 0 \\ 0 \\ \mp(1+z\bar{z})^{-2}A_{\pm}(z) \end{pmatrix}$$

These solutions are self and anti-self-dual respectively under the operation of i times the Hodge Star. The factor of i comes from the alteration in the definition of the Hodge Star when the exterior algebra is taken to be complex rather than real. This is due to the fact that if $z = x + iy$, $dz \wedge d\bar{z} = -2idx \wedge dy$.

Acting with the Hamiltonian H_s on $\phi_{\pm}(z)$ gives for the (0,0)-form piece, the equation

$$-(1+z\bar{z})^2\frac{d^2A_{\pm}(z)}{dzd\bar{z}} + \frac{s^2z\bar{z}}{(1+z\bar{z})^2}A_{\pm}(z) \mp s\frac{(1-z\bar{z})}{1+z\bar{z}}A_{\pm}(z) = EA_{\pm}(z) \qquad (2).$$

The equation for the (1,1)-form piece is exactly the same

$$-\frac{d^2A_{\pm}(z)}{dzd\bar{z}} + \frac{s^2z\bar{z}}{(1+z\bar{z})^4}A_{\pm}(z) \mp s\frac{(1-z\bar{z})}{(1+z\bar{z})^3}A_{\pm}(z) = \frac{EA_{\pm}(z)}{(1+z\bar{z})^2} \quad .$$

So solving the Schrödinger Equation on even-forms reduces to finding solutions $A \pm (z)$ of this equation. The odd-form solutions can then be found by applying the supersymmetry operators.

The excited states have an eight-fold degeneracy; $\phi_+(z)$ and $\phi_-(z)$ are contained in separate supersymmetry quadruplets related by Poincare duality. The other members of the quadruplets may be found by acting with the supersymmetry operators

$$\psi^1_+(z) = Q_{s1}\phi_+(z) = \sqrt{2} i^{\frac{1}{2}} \begin{pmatrix} 0 \\ \frac{dA_+(z)}{dz} + \frac{s\bar{z}}{(1+z\bar{z})^2} A_+(z) \\ 0 \\ 0 \end{pmatrix}$$

$$\psi^2_+(z) = Q_{s2}\phi_+(z) = \sqrt{2} i^{-\frac{1}{2}} \begin{pmatrix} 0 \\ 0 \\ \frac{dA_+(z)}{d\bar{z}} + \frac{sz}{(1+z\bar{z})^2} A_+(z) \\ 0 \end{pmatrix}$$

$$\phi^{1,2}_+(z) = Q_{s1}Q_{s2}\phi_+(z) = -i \begin{pmatrix} 1 \\ 0 \\ 0 \\ i(1+z\bar{z})^{-2} \end{pmatrix} [E - \frac{2s^2 z\bar{z}}{(1+z\bar{z})^2} - s[z\frac{d}{dz} + \bar{z}\frac{d}{d\bar{z}}]]A_+(z)$$

so the quadruplet contains an anti-self-dual even-form as well as a self-dual even-form. The other quadruplet which contains $\phi_-(z)$ is obtained from this one by exchanging $s \longrightarrow -s$ and $A_+(z) \longrightarrow A_-(z)$. As s tends to zero $A_+(z)$ becomes equal to $A_-(z)$ and the quadruplets converge to become identical.

It is now obvious what the zero energy eigensolutions of H_s are as these must be annihilated by Q_{s1} and Q_{s2} . For this to occur either $A_+(z)$ must be real, satisfying

$$\frac{dA_+(z)}{dz} + \frac{s\bar{z}}{(1+z\bar{z})^2} A_+(z) = 0$$

and the complex conjugate equation or $A_-(z)$ must be real and satisfy

$$\frac{dA_-(z)}{dz} - \frac{s\bar{z}}{(1+z\bar{z})^2} A_-(z) = 0 \qquad .$$

Thus there are two zero energy eigensolutions

$$\phi^0_+(z) = \exp[\frac{+s}{1+z\bar{z}}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i(1+z\bar{z})^{-2} \end{pmatrix}$$

which is self-dual and

$$\phi^0_-(z) = \exp[\frac{-s}{1+z\bar{z}}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ +i(1+z\bar{z})^{-2} \end{pmatrix}$$

which is anti-self-dual. As s tends to zero these solutions tend to the sum and difference of the representatives of the cohomology of the 2-sphere. The fact that the zero energy solutions still exist means that supersymmetry is unbroken; unlike the case of the 2-torus, see Appendix 3(iii), where the introduction of a Killing Vector removes all zero energy states.

The zero energy solutions are the only ones that can be found exactly. To study the spectrum of the excited states approximate methods must be used.

The equations for $A_{\pm}(z)$ may be further simplified by using the zero energy solutions as integrating factors. Substituting

$$A_{\pm}(z) = \chi(z)\exp[\frac{\pm s}{1+z\bar{z}}]$$

into equation (4.1.2) gives immediately

$$-(1+z\bar{z})^2\frac{d^2\chi(z)}{dzd\bar{z}} \pm s[z\frac{d\chi(z)}{dz} + \bar{z}\frac{d\chi(z)}{d\bar{z}}] = E\chi(z) \qquad (3)$$

where the Schrödinger Equation now takes the form of the Laplacian plus a term proportional to the operator $z\frac{d}{dz} + \bar{z}\frac{d}{d\bar{z}}$. This vector commutes with the Killing Vector $\mathbf{k} = i[z\frac{d}{dz} - \bar{z}\frac{d}{d\bar{z}}]$, but not with the other two generators of SU(2), so m, the eigenvalue of I_3 , is still a good quantum number, but the full SU(2) symmetry has been broken to U(1).

There are two sets of excited states which correspond to excitations around the two zero energy ground states. The signs of the parameter s in equation (3) for these two sets of states are opposite, however it turns out that the energy levels are the same in both cases.

4.3 Perturbation Theory

For small s the term $s[z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}}]$ in the Schrödinger Equation may be treated as a perturbation of Legendre's Equation. This vector can be rewritten so that its effect on Associated Legendre Functions is much more obvious.

$$z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}} = \frac{z}{1+z\bar{z}}[\frac{d}{dz}+\bar{z}^2\frac{d}{d\bar{z}}]+\frac{\bar{z}}{1+z\bar{z}}[z^2\frac{d}{dz}+\frac{d}{d\bar{z}}]$$

$$= \sqrt{2}Y_1^1(z)I_- - \sqrt{2}Y_1^{-1}(z)I_+$$

$Y_1^{\pm 1}(z)$ are the Associated Legendre Functions with $n=1, m=\pm 1$ and $I\pm$ are the SU(2) raising and lowering operators. The effect of this operator is therefore to conserve m and change n by plus or minus one. This indicates that the first order shift in energy

$$\triangle^{(1)}(z)E_n^m = s\frac{< Y_n^m(z,\bar{z})|z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}}|Y_n^m(z,\bar{z}) >}{< Y_n^m(z,\bar{z})|Y_n^m(z,\bar{z}) >}$$

is zero for all n and m. In terms of the coordinates on the 2-sphere the vanishing of $\triangle^{(1)}(z)E_n^m$ is due to the fact that the perturbation is anti-symmetric under the change of coordinates $z \longrightarrow w = \frac{1}{z}$, corresponding to the inversion of the sphere

$$z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}} = -[w\frac{d}{dw}+\bar{w}\frac{d}{d\bar{w}}] \qquad .$$

This is due to the opposite direction of the rotation around the sphere's equation when viewed from the other pole. Applying this change of coordinates to the matrix elements gives

$$< Y_n^m(z,\bar{z})|z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}}|Y_n^m(z,\bar{z}) >= - < Y_n^{-m}(w,\bar{w})|w\frac{d}{dw}+\bar{w}\frac{d}{d\bar{w}}|Y_n^{-m}(w,\bar{w}) >$$

which after taking the complex conjugate of the right hand side implies that the matrix elements must vanish. Thus all odd orders of perturbation theory are zero, so the sign of s is irrelevant and the excitations around the two different ground states are degenerate to all orders in perturbation theory. Each excited state is a member of a quadruplet, so the total degeneracy due to supersymmetry is eight-fold.

4.3.1 Second Order Perturbation Theory

In the formula for the second order perturbation theory shift in energy

$$\triangle^{(2)}(z)E_n^m = s^2 \sum_{n',m',n'\neq n,m'\neq m} \frac{< Y_{n'}^{m'}(z,\bar{z})|z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}}|Y_n^m(z,\bar{z}) >< Y_n^m(z,\bar{z})|z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}}|Y_{n'}^{m'}(z,\bar{z}) >}{(E_n - E_{n'}) < Y_{n'}^{m'}(z,\bar{z})|Y_{n'}^{m'}(z,\bar{z}) >< Y_n^m(z,\bar{z})|Y_n^m(z,\bar{z}) >}$$

which is in principle an infinite sum. Only the two terms $n' = n+1$, $n' = n-1$ with $m' = m$ contribute. The easiest way to calculate the shift is to use a formula from the theory of Legendre Functions

$$sin\theta\frac{d}{d\theta}P_n^m(\theta,\phi) = \frac{n(n-|m|+1)}{2n+1}P_{n+1}^m(\theta,\phi) - \frac{(n+|m|)(n+1)}{2n+1}P_{n-1}^m(\theta,\phi) \qquad ,$$

where $P_n^m(\theta,\phi) = Y_n^m(z,\bar{z})$.

In terms of these coordinates $z = tan(\frac{\theta}{2})e^{i\phi}$, where θ and ϕ are the longitude and latitude on the sphere, the perturbation takes the form

$$z\frac{d}{dz} + \bar{z}\frac{d}{d\bar{z}} = sin(\theta)\frac{d}{d\theta} \qquad .$$

The effect of this operator on the Associated Legendre Functions $Y_n^m(z,\bar{z})$ is therefore

$$[z\frac{d}{dz} + \bar{z}\frac{d}{d\bar{z}}]Y_n^m(z,\bar{z}) = \frac{n(n-|m|+1)}{2n+1}Y_{n+1}^m(z,\bar{z}) - \frac{(n+|m|)(n+1)}{2n+1}Y_{n-1}^m(z,\bar{z}) \tag{1}$$

Substituting this formula into the expression for the second order energy shift

$$\triangle^{(2)}E_n^m = s^2[\frac{< Y_{n+1}^m(z,\bar{z})|z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}}|Y_n^m(z,\bar{z}) >< Y_n^m(z,\bar{z})|z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}}|Y_{n+1}^m(z,\bar{z}) >}{(n(n+1)-(n+1)(n+2)) < Y_{n+1}^m(z,\bar{z})|Y_{n+1}^m(z,\bar{z}) >< Y_n^m(z,\bar{z})|Y_n^m(z,\bar{z}) >}$$

$$+\frac{< Y_{n-1}^m(z,\bar{z})|z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}}|Y_n^m(z,\bar{z}) >< Y_n^m(z,\bar{z})|z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}}|Y_{n-1}^m(z,\bar{z}) >}{(n(n+1)-n(n-1)) < Y_{n-1}^m(z,\bar{z})|Y_{n-1}^m(z,\bar{z}) >< Y_n^m(z,\bar{z})|Y_n^m(z,\bar{z}) >}]$$

gives

$$\triangle^{(2)}E_n^m = s^2[\frac{1}{-2(n+1)}[\frac{(n-|m|+1)}{2n+1}][\frac{-n(n+1+|m|)(n+2)}{2n+3}]$$

$$+\frac{1}{2n}[\frac{-(n+|m|)(n+1)}{2n+1}][\frac{(n-1)(n-|m|)}{2n-1}]]$$

$$= s^2\left[\frac{n(n+2)(n+1-|m|)(n+1+|m|)}{(2n+1)(2n+2)(2n+3)} - \frac{(n-1)(n+1)(n-|m|)(n+|m|)}{2n(2n-1)(2n+1)}\right] \quad .$$

This formula depends on $|m|$, so the SU(2) (2n+1)-plets are split into n doublets and a singlet.

On the lowest of the excited representations, the **3** which has energy $E = 2$ and the **5** which has energy $E = 6$, the second order energy changes due to the perturbation are

		$Y_1^{-1}(z)$	$Y_1^0(z)$	$Y_1^1(z)$	
$\triangle^{(2)}E_1$		$\frac{3}{20}s^2$	$\frac{1}{5}s^2$	$\frac{3}{20}s^2$	
	$Y_2^{-2}(z)$	$Y_2^{-1}(z)$	$Y_2^0(z)$	$Y_2^1(z)$	$Y_2^2(z)$
$\triangle^{(2)}E_2$	$\frac{4}{21}s^2$	$\frac{13}{84}s^2$	$\frac{1}{7}s^2$	$\frac{13}{84}s^2$	$\frac{4}{21}s^2$

4.3.2 Fourth Order Perturbation Theory

The next non-zero order of perturbation theory is fourth order. The corrections to the energy to this order may be found by a substitution of the form of

$$\chi_n^m(z) = Y_n^m(z) + s[\alpha Y_{n-1}^m(z) + \beta Y_{n+1}^m(z)] + s^2[\gamma Y_{n-2}^m(z) + \delta Y_{n+2}^m(z)]$$

into equation (3) of section 4.2 and use of formula (1) of section 4.3.1. The energy to this order is

$$E = n(n+1) + ks^2 + qs^4 \quad .$$

Equating coefficients of $Y_{n\pm2}^m(z)$, $Y_{n\pm1}^m(z)$ and $Y_n^m(z)$ in this equation, in the case where $m = 0$, leads to the expression

$$\triangle^{(4)}E_n^0 = qs^4 = \frac{s^4}{8}\left[\frac{(n-2)(n-1)^3 n(n+1)}{(2n-3)(2n-1)^3(2n+1)} - \frac{n(n+1)(n+2)^3(n+3)}{(2n+1)(2n+3)^3(2n+5)}\right]$$

$$+\triangle^{(2)}E_n^0\frac{s^2}{4}\left[\frac{(n-1)(n+1)}{(2n-1)(2n+1)} + \frac{n(n+2)}{(2n+1)(2n+3)}\right]$$

Substituting in the value $n = 1$, for the first excited state with m = 0, gives the fourth order energy shift,

$$\triangle^{(4)}E_1^0 = \frac{-1}{3500}s^4$$

The next case n = 2, gives

$$\triangle^{(4)}E_2^0 = \frac{-1}{4116}s^4$$

4.4 Asymptotic Solutions

For large s the spectrum of the Schrödinger Equation can be found as an asymptotic expansion in $\frac{1}{s}$. The low energy solutions are localized around the fixed points of the vector $z\frac{d}{dz}+\bar{z}\frac{d}{d\bar{z}}$, which are where the Hamiltonian is smallest. These fixed points are the same as those of I_3, $z=0$ and $z=\infty$, the south and north poles of the sphere.

4.4.1 The Harmonic Oscillator Approximation

In the large s limit the Hamiltonian tends to a supersymmetric harmonic oscillator around these fixed points. Substituting $v=\sqrt{s}z$ into equation (1) of section 4.2 we obtain

$$-[1+\frac{v\bar{v}}{s}]^2\frac{d^2\psi(v)}{dvd\bar{v}}+\frac{v\bar{v}}{[1+\frac{v\bar{v}}{s}]^2}\psi(v)+i\begin{pmatrix} 0 & 0 & 0 & -[1-\frac{v\bar{v}}{s}][1+\frac{v\bar{v}}{s}] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{[1-\frac{v\bar{v}}{s}]}{[1+\frac{v\bar{v}}{s}]^3} & 0 & 0 & 0 \end{pmatrix}\psi(v)$$

$$=\frac{E}{s}\psi(v)$$

Taking s large and expanding in powers of $\frac{1}{s}$ gives the approximate form of the Schrödinger Equation near the south pole

$$-[1+\frac{v\bar{v}}{s}]^2\frac{d^2\psi(v)}{dvd\bar{v}}+[1-\frac{2v\bar{v}}{s}+\cdots]v\bar{v}\psi(v)+i\begin{pmatrix} 0 & 0 & 0 & -[1-\frac{(v\bar{v})^2}{s^2}] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1-\frac{4v\bar{v}}{s}+\cdots & 0 & 0 & 0 \end{pmatrix}\psi(v)$$

$$=\frac{E}{s}\psi(v)$$

Neglecting terms of order $\frac{1}{s}$ and smaller and substituting $z=\frac{v}{\sqrt{s}}$ back into the equation gives the harmonic oscillator equation

$$-\frac{d^2\psi(z)}{dzd\bar{z}} + s^2 z\bar{z}\psi(z) + is \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \psi(z) = \tilde{E}\psi(z) \qquad .$$

The coordinate z is not well defined at the other fixed point of $\mathbf{k}$, so the second coordinate patch must be utilised. Using the coordinate $w = \frac{1}{z}$, the Schrödinger Equation becomes

$$-(1+w\bar{w})^2 \frac{d^2\psi(w)}{dwd\bar{w}} + s^2 \frac{w\bar{w}}{(1+w\bar{w})^2}\psi(w) + is \begin{pmatrix} 0 & 0 & 0 & -(w\bar{w}-1)(1+w\bar{w}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{(w\bar{w}-1)}{(1+w\bar{w})^3} & 0 & 0 & 0 \end{pmatrix} \psi(w)$$

$$= E\psi(w)$$

where a factor of $(w\bar{w})^2$ has been inserted into the off-diagonal matrix to compensate for the transformation of the two-form $dz \wedge d\bar{z} = (w\bar{w})^2 dw \wedge d\bar{w}$. For small w, near the north pole, this equation takes the same form as that near the south pole apart from the alteration $s \longrightarrow -s$, due to the opposite manner of the rotation around the equator when the sphere is viewed from the opposite pole.

$$\mathbf{k} = is[z\frac{d}{dz} - \bar{z}\frac{d}{d\bar{z}}] = -is[w\frac{d}{dw} - \bar{w}\frac{d}{d\bar{w}}] = is[\bar{w}\frac{d}{d\bar{w}} - w\frac{d}{dw}]$$

To write the Killing Vector in the standard form, where it is multiplied by a positive parameter, interchanges the coordinates w and $\bar{w}$, so that the coordinates in the second patch have opposite orientation to those in the first. This explains why the zero energy solutions near the two poles have opposite duality. The zero energy solution near the south pole is self-dual

$$\psi_0^S(z) = \exp[-sz\bar{z}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i \end{pmatrix}$$

whilst that near the north pole is anti-self-dual

$$\psi_0^N(z) = \exp[-sw\bar{w}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix} \qquad .$$

The eight-fold degeneracy of the excited states is now realised as a degeneracy between two quadruplets, one at each pole. The general form of the excited states can be found using harmonic oscillator ladder operators. At the sound pole the solutions are

$$S^{+}_{p,q}(z) = [\frac{d}{d\bar{z}} - sz]^{p}[\frac{d}{dz} - s\bar{z}]^{q} \exp[-sz\bar{z}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i \end{pmatrix}$$

with energy $E = (p+q)s$ and with the eigenvalue of the Lie Derivative along the Killing Vector $\mathcal{L}_{\mathbf{k}}$, $m = (p-q)$. As these states are harmonic oscillator eigenstates, they form SU(2) representations of dimension $(\hat{n}+1)$ where $\hat{n} = p+q$. When the $O(1)$ corrections to the energy are included this SU(2) symmetry is broken to U(1).

To this approximation the supersymmetry operators act on the even-form solutions $S^{+}_{p,q}(z)$ with harmonic oscillator lowering operators to give the odd-form solutions as

$$\tilde{Q}_{s1} S^{+}_{p,q}(z) = T^{1}_{p,q-1}(z) = [\frac{d}{d\bar{z}} - sz]^{p}[\frac{d}{dz} - s\bar{z}]^{q-1} \exp[-sz\bar{z}] \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\tilde{Q}_{s2} S^{+}_{p,q}(z) = T^{2}_{p-1,q}(z) = [\frac{d}{d\bar{z}} - sz]^{p-1}[\frac{d}{dz} - s\bar{z}]^{q} \exp[-sz\bar{z}] \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

with energy $E = (p+q)s$ and $m = p - q \pm 1$ respectively. Both sets of states have a (p+q)-fold degeneracy. The other even-form states in the supersymmetry multiplets, obtained by acting on $S^{+}_{p,q}(z)$ with both $\tilde{Q}_{s1}$ and $\tilde{Q}_{s2}$ consecutively are

$$S^{-}_{p,q}(z) = [\frac{d}{d\bar{z}} - sz]^{p-1}[\frac{d}{dz} - s\bar{z}]^{q-1} \exp[-sz\bar{z}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix}$$

where $E = (p+q)s$ and $m = (p-q)$ and the degeneracy is $(p+q-1)$ - fold.

The total degeneracy of even-form solutions of energy $E = \hat{n}s$ localized around the south pole, as for odd-forms, is

$$(p+q+1) + (p+q-1) = 2(p+q) = 2\hat{n} \; .$$

The results near the north pole are completely analogous.

4.4.2 $O(1)$ Asymptotic Corrections

The second order terms in the asymptotic expansion of the energy are of order one and so don't vanish in the large s limit. The easiest way to calculate these corrections to the harmonic oscillator energies is to use the form of the Schrödinger Equation, equation (3) of section 4.2

$$-(1+z\bar{z})^2\frac{d^2\chi(z)}{dzd\bar{z}}+s[z\frac{d\chi(z)}{dz}+\bar{z}\frac{d\chi(z)}{d\bar{z}}]=E\chi(z)$$

The plus sign is taken corresponding to excitations around the self-dual ground state, which is localized around the south pole for large 2. When considering solutions localized around the north pole the sign in the equation is again positive because the minus sign corresponding to the anti-self-dual ground state is cancelled by the fact that $s\longrightarrow -s$ when using the coordinate around the north pole $w=\frac{1}{z}$. Because of this the $O(1)$ corrections are the same at both poles.

Putting $v=\sqrt{s}z$ and keeping terms up to $O(1)$ gives the equation

$$-[1+\frac{2v\bar{v}}{s}]s\frac{d^2\chi(v)}{dvd\bar{v}}+s[v\frac{d\chi(v)}{dv}+\bar{v}\frac{d\chi(v)}{d\bar{v}}]=\tilde{E}\chi(v) \qquad .$$

Inserting $\chi(v)$ as an expansion in powers of v and $\bar{v}$

$$\chi(v)=\sum_{p,q\geq 0}a_{p,q}v^p\bar{v}^q$$

gives

$$-[1+\frac{2v\bar{v}}{s}]\sum_{p,q\geq 0}pqa_{p,q}v^{p-1}\bar{v}^{q-1}+\sum_{p,q\geq 0}a_{p,q}(p+q)v^p\bar{v}^q=\frac{\tilde{E}}{s}\sum_{p,q\geq 0}a_{p,q}v^p\bar{v}^q$$

which implies the following equation for the coefficients $a_{p,q}$ in the solutions

$$-(p+1)(q+1)a_{p+1,q+1}-\frac{2}{s}pqa_{p,q}+(p+q)a_{p,q}=\frac{\tilde{E}}{s}a_{p,q}$$

For a closed series solution $a_{p+1,q+1}=0$, which gives the energy to $O(1)$ as

$$\tilde{E}=(p+q)s-2pq \qquad .$$

The $(\hat{n}+1)$ - fold degeneracy of the harmonic oscillator has been split, so that states either form doublets, produced by interchanging p and q in the formula for $\chi(v)$, or singlets if $p=q$. This is due to the U(1) symmetry of H_s. There is of course still the eight-fold degeneracy due to supersymmetry as well. The odd-form solutions have energies $\tilde{E}=(p+q+1)s-2(p+1)q$ and $\tilde{E}=(p+q+1)s-2p(q+1)$ and the other even-form solutions have $\tilde{E}=(p+q+2)s-2(p+1)(q+1)$.

The spectrum of the low lying states in this approximation and the degeneracy of the even-form states near the south pole is the following.

$energy$	$degeneracy$	$values.of$	$p-and-q$		
		$self-dual$	$even-forms$	$anti-selfdual$	$even-forms$
0	1	$p=0,$	$q=0$		
s	2	$p=0,$	$q=1$		
		$p=1,$	$q=0$		
$2s-2$	2	$p=1,$	$q=1$	$p=0,$	$q=0$
$2s$	2	$p=0,$	$q=2$		
		$p=2,$	$q=0$		
$3s-4$	4	$p=2,$	$q=1$	$p=1,$	$q=0$
		$p=1,$	$q=2$	$p=0,$	$q=1$
$3s$	2	$p=0,$	$q=3$		
		$p=3,$	$q=0$		
$4s-8$	2	$p=2,$	$q=2$	$p=1,$	$q=1$
$4s-6$	4	$p=3,$	$q=1$	$p=2,$	$q=0$
		$p=1,$	$q=3$	$p=0,$	$q=2$
$4s$	2	$p=4,$	$q=0$		
		$p=0,$	$q=4$		

4.5 Summary

After the inclusion of the Killing Vector the asymptotic zero energy solutions localized one near each fixed point, are in a one-to-one correspondence with the exact zero energy solutions. Near the south pole $z = 0$

$$\phi_+^0(z) = \exp[\frac{+s}{1+z\bar{z}}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i(1+z\bar{z})^{-2} \end{pmatrix} \approx \exp(s)\exp[-sz\bar{z}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i \end{pmatrix}$$

and near the north pole $w = \frac{1}{z} = 0$

$$\phi_-^0(z) = \exp[\frac{-s}{1+z\bar{z}}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i(1+z\bar{z})^{-2} \end{pmatrix} = \exp[\frac{-sw\bar{w}}{1+w\bar{w}}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ i(1+w\bar{w}) \end{pmatrix} \approx \exp[-sw\bar{w}] \begin{pmatrix} 1 \\ 0 \\ 0 \\ +i \end{pmatrix} .$$

This correspondence is due to Witten's relation $n_+ = N_+$.

By the index arguments of section 3, the number of even-form solutions minus the number of odd-form solutions must be independent of s. For $s = 0$ on the 2-sphere, the zero energy solutions corresponding to the representatives of the cohomology are the constant (0,0)-form and the Kähler form $i(1+z\bar{z})^{-2}dz \wedge d\bar{z}$, both of which are even-forms, therefore $\chi(S^2) = 2$. For non-zero s there are again two even-form zero energy solutions and no odd-form ones and ultimately in the large s limit these solutions become localized one at each fixed point. This corresponds to the Lefschetz Fixed Point Theorem,

$$\chi(M) = M_+ - M_- = \text{ number of fixed point } = N_+ = 2 \qquad .$$

The excited states, which form the (2n+1)-dimensional representations of SU(2) when s = 0 are more difficult to analyse for non-zero s, but approximate methods show how the SU(2) symmetry breaks to U(1).

Appendix 4(ii)

Computer Analysis

After the introduction of the Killing Vector iI_3 into the supersymmetry algebra, the Schrödinger Equation (3) of section 4.2 is not soluble analytically for excited states, so a computer program was written to find accurate values of the eigenvalues of this equation for all values of s. The program uses the library subroutine D02KDF from the NAG FORTRAN library, which finds a specified eigenvalue E of a Sturm-Liouville system defined by a second order self-adjoint differential equation

$$\frac{d}{d\theta}[P(\theta)\frac{d\chi(\theta)}{d\theta}] + Q(\theta;E)\chi(\theta) = 0 \quad , \qquad a < \theta < b$$

together with boundary conditions at the endpoints a and b .

To transform equation (4.1.3) into a suitable form for the computer analysis the substitution

$$z = tan[\frac{\theta}{2}]e^{i\phi} \qquad \text{where} \quad 0 \le \theta < \pi \quad , \quad 0 \le \phi < 2\pi$$

was used leading to the equation,

$$-[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}[sin\theta\frac{\partial}{\partial\theta}] + \frac{1}{sin^2\theta}\frac{\partial}{\partial\phi^2}]\chi(\theta) + s[sin\theta\frac{\partial\chi(\theta)}{\partial\theta}] = E\chi(\theta)$$

which is the Associated Legendre's Equation plus the s-dependent term.

The operator $i\frac{\partial}{\partial\phi}$ is equivalent to the central charge P and commutes with the Hamiltonian, so the solutions of the Schrödinger Equation may be taken to be eigensolutions of this operator with eigenvalue m, thus reducing the equation to an ordinary differential equation. The self-adjoint form necessary for specification in the library subroutine is given by

$$\frac{-1}{sin\theta\exp^{s[cos\theta]}}\frac{d}{d\theta}[sin\theta\exp^{s[cos\theta]}\frac{d\chi(\theta}{d\theta}] + \frac{m^2\chi(\theta)}{sin^2\theta} = E\chi(\theta)$$

so that the functions $P(\theta)$ and $Q(\theta;E)$ are

$$P(\theta) = -sin\theta\exp[scos\theta]$$

$$Q(\theta;E) = -sin\theta\exp[scos\theta][\frac{m^2}{sin^2\theta} - E]$$

The boundary conditions are specified in YL(1) and YL(2) by the values of $\chi(\theta)$ and $P(\theta)\frac{d\chi(\theta)}{d\theta}$ at the other end point, only the ratios of the two values being important.

This is the program used to find the eigenvalues of the Schrödinger Equation on S^2 after the introduction of the central charge.

```
PROGRAM         SUSYS2
REAL *8         DELAM, ELAM, TOL, PI, D, HMAX(2, 4), XPOINT(4), M, S
INTEGER         IFAIL, K, MAXIT
EXTERNAL BDYVL, COEFF, MONIT
```

```
COMMON          PI, D, M, S
PRINT *,        'ENTER M, S, K, ELAM'
READ *,         M, S, K, ELAM
D = 0.001
TOL = 0.0001
DELAM = 0.1
PI = X01AAF(D)
MAXIT = 0
XPOINT(1) = 0.0
XPOINT(2) = 0.001
XPOINT(3) = PI-0.001
XPOINT(4) = PI
HMAX(1, 1) = 0.0
IFAIL = 0
CALL  D02KDF  (XPOINT, 4, COEFF, BDYVL, K, TOL, ELAM,
          DELAM, HMAX, MAXIT, 0, MONIT, IFAIL)
WRITE (*, *)  K, ELAM, DELAM, IFAIL
STOP
END
SUBROUTINE MONIT (MAXIT, IFLAG, ELAM, FINFO)
INTEGER         MAXIT, IFLAG, I
REAL  *8        ELAM, FINFO (15), PI, D, M, S
COMMON          PI, D, M, S
WRITE (*, *)    MAXIT, IFLAG, ELAM, (FINFO(I), I = 1, 4)
RETURN
END
SUBROUTINE  BDYVL (XL, XR, ELAM, YL, YR)
REAL  *8        ELAM, XL, XR, PI, D, YL(3), YR(3), M, S
COMMON          PI, D, M, S
YL(1) = 1.0
YL(2) = M*EXP(S)
YR(1) = 1.0
YR(2) = -M*EXP(-S)
RETURN
END
SUBROUTINE COEFF (P, Q, DQDL, X, ELAM, JINT)
REAL  *8        P, Q, M, S, DQDL, X, ELAM, PI, D
INTEGER         JINT
```

```
COMMON      PI, D, M, S
P        = -SIN(X)*EXP(S*COS(X))
Q        = -P*(M*M/(SIN(X)*SIN(X)) – ELAM)
DQDL     = P
RETURN
END
```

The numerical results from the computer show how the energies of the eigenstates as functions of the parameter s interpolates between the perturbative regime $s \ll \frac{1}{4}$ and the asymptotic regime $s >> 2$, and verify the approximate analysis in these two cases.

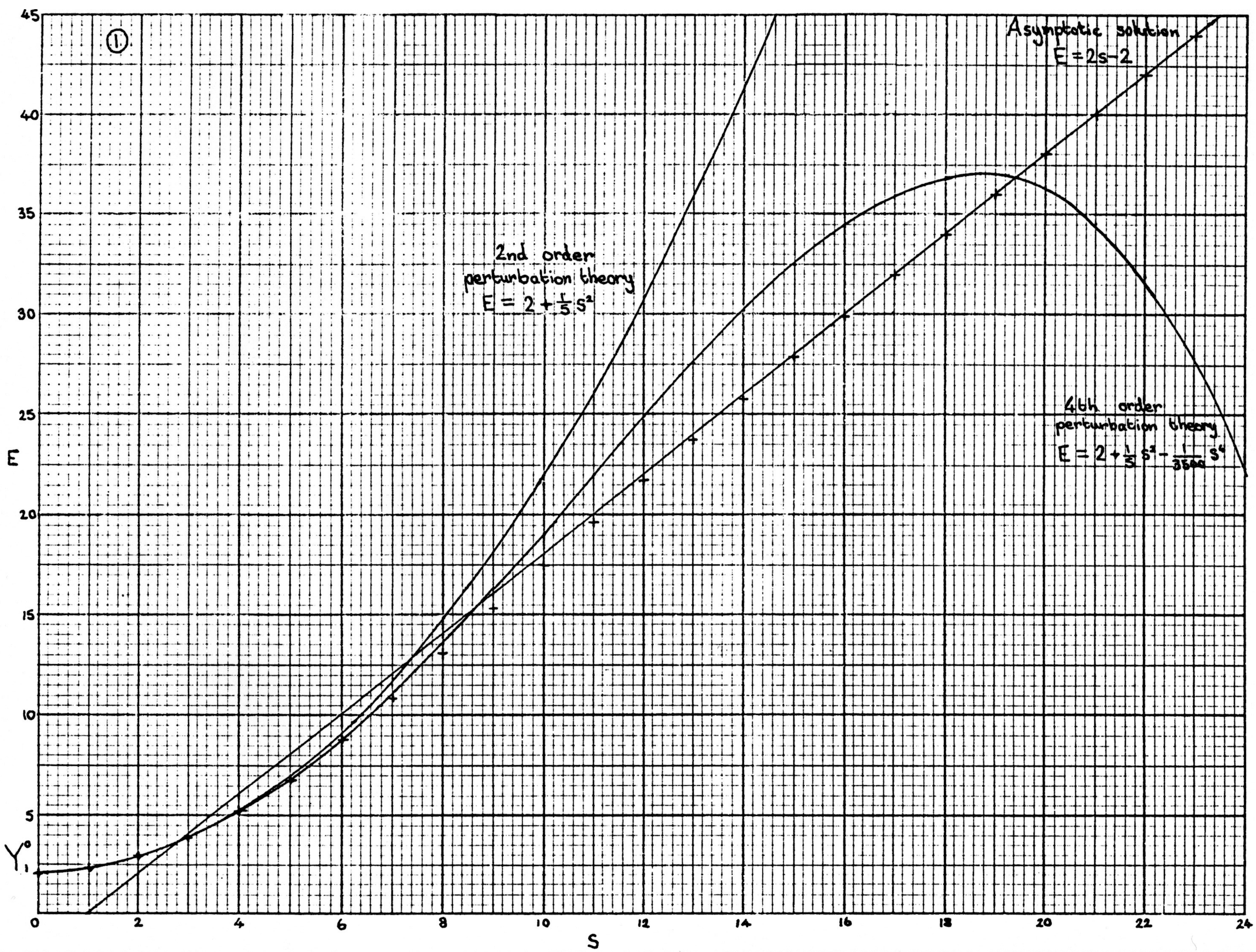
The first two graphs compare the results from the computer analysis with second order and fourth order perturbation theory and also with the asymptotic results in the cases of the first two excited states with $m = 0$. These graphs show a good agreement between the perturbative results and the numerical values from the computer for surprisingly large values of s.

The third graph shows how the spectrum varies as s increases for the low lying energy levels. The values of l on the right-hand side of the graph indicate the level of the corresponding Legendre Funciton Y_l^m which the eigensolution tends to as s tends to zero. The graph shows the symmetry breaking which splits the SU(2) $(2l+1)$-plets into l doublets and a singlet.

Asymptotically the graphs have gradients proportional to s multiplied by an integer corresponding to the harmonic oscillator energies $E = \hat{n}s$, with a degeneracy of $2\hat{n}$. Taking into account the $O(1)$ corrections the graphs show agreement with the energy levels given at the end of section 4.

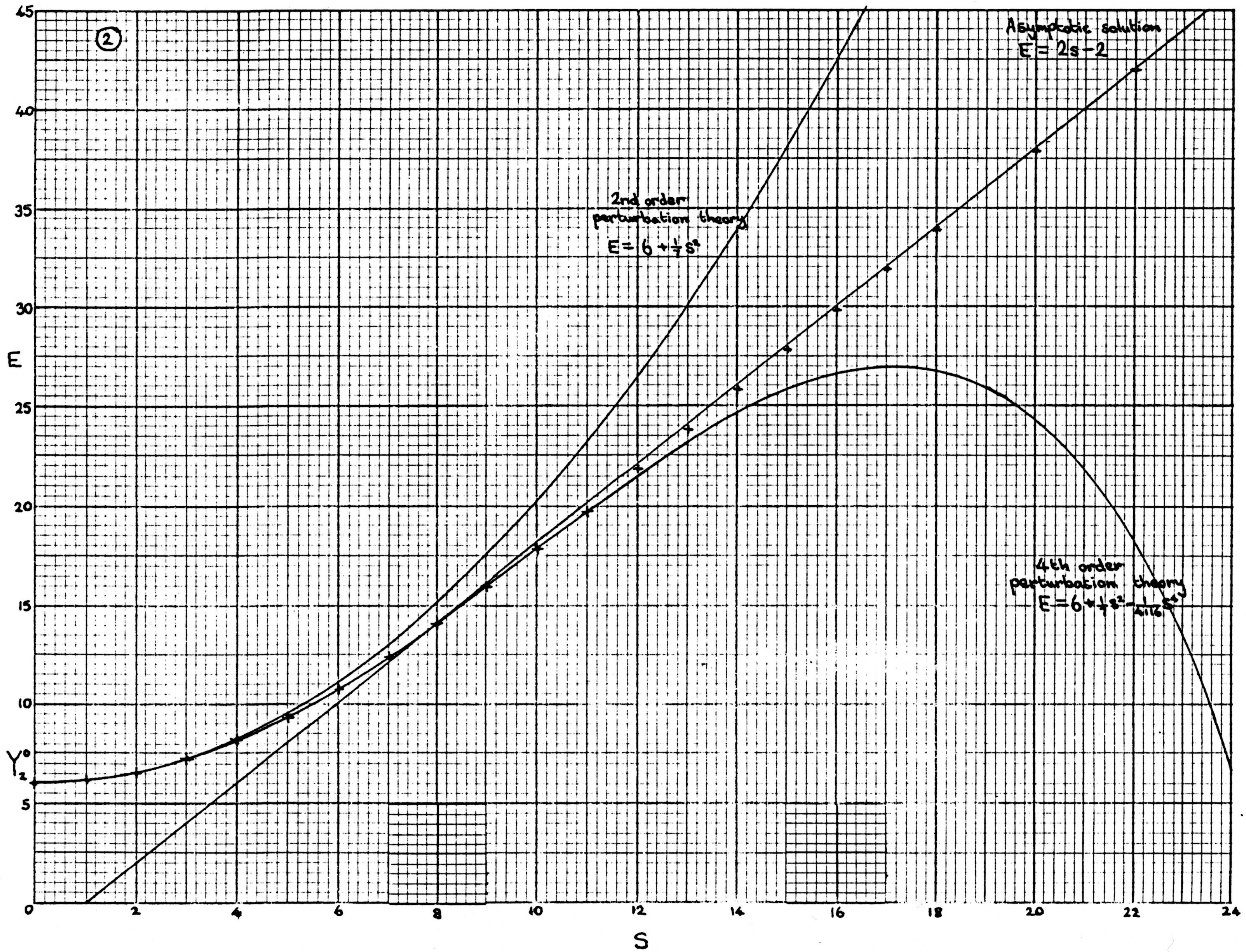
Graph 3, for $s \neq 0$, in fact only shows half the even-form solutions. Apart from the zero energy solutions each even-form solution is paired with an odd-form solution.

Graph 1 on the following page shows the energy of the first excited state with m = 0 for different values of the parameter s calculated using the various approximate methods. The crosses mark the results of the computer analysis.

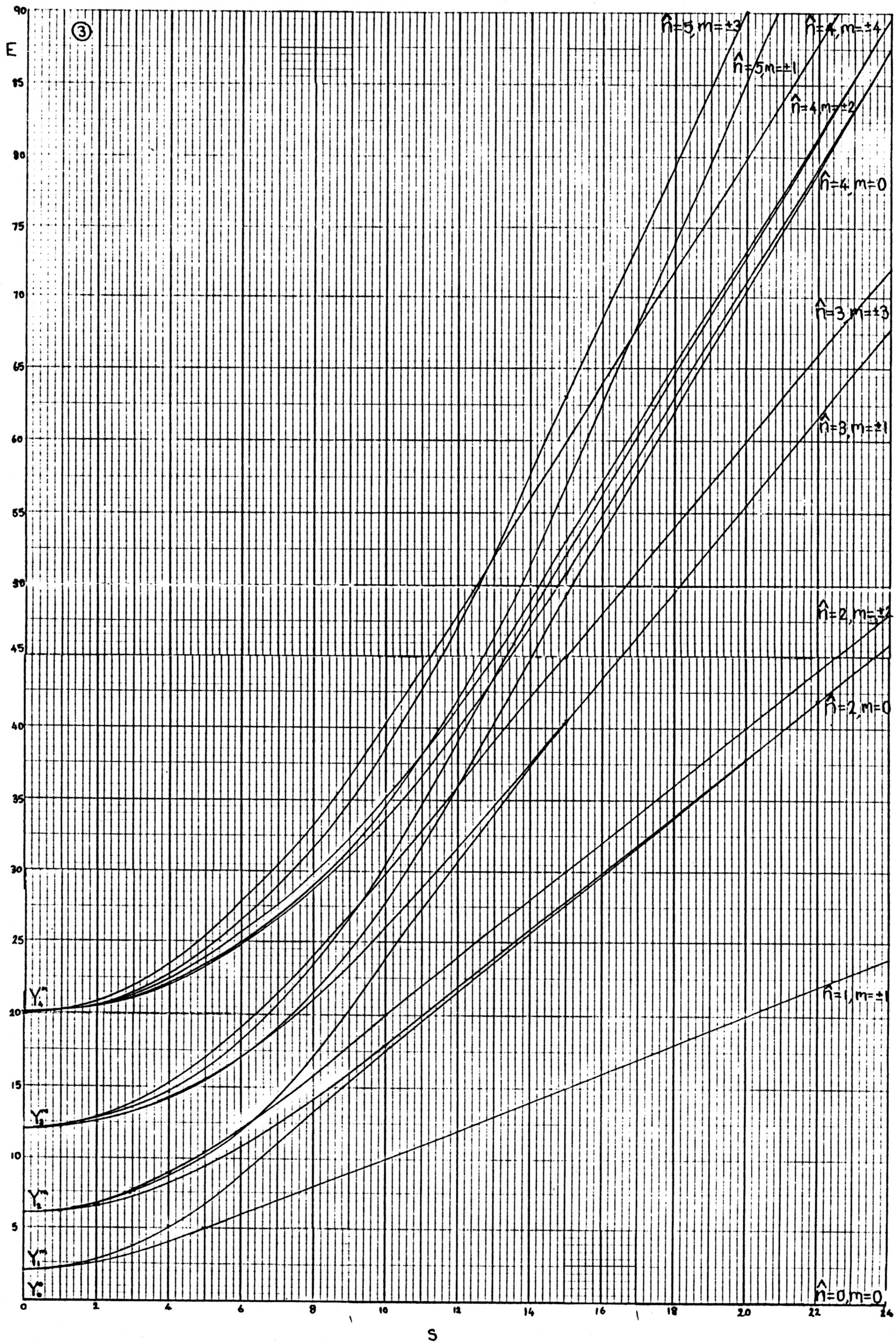


1
Asymptotic solution
E = 2s - 2
2nd order
perturbation theory
E = 2 + 1/5 s²
4th order
perturbation theory
E = 2 + 1/5 s² - 1/3500 s⁴
E
S
Y°₁
0
2
4
6
8
10
12
14
16
18
20
22
24
5
10
15
20
25
30
35
40
45

Graph 2 on the following page shows the energy of the second excited state with $m = 0$ for different values of the parameter s calculated using the various approximate methods. The crosses mark the results of the computer analysis.


2
Asymptotic solution
E = 2s - 2
2nd order
perturbation theory
E = 6 + 1/7 s²
4th order
perturbation theory
E = 6 + 1/7 s² - 1/4116 s⁴
E
S
Y°₂
0
5
10
15
20
25
30
35
40
45
2
4
6
8
10
12
14
16
18
20
22
24

Graph 3 on the following page shows the lowest energy levels as functions of s as given by the computer analysis. When $s = 0$ at the left-hand side the solutions are Associated Legendre Functions Y_n^m, on the right the energies tend to the asymptotic vales, $\hat{n}$ is the level of the harmonic oscillator solution.


③
E
S
$\hat{n}=5, m=\pm3$
$\hat{n}=4, m=\pm4$
$\hat{n}=5, m=\pm1$
$\hat{n}=4, m=\pm2$
$\hat{n}=4, m=0$
$\hat{n}=3, m=\pm3$
$\hat{n}=3, m=\pm1$
$\hat{n}=2, m=\pm2$
$\hat{n}=2, m=0$
$\hat{n}=1, m=\pm1$
$\hat{n}=0, m=0$
Y_4^m
Y_3^m
Y_2^m
Y_1^m
Y_0^0