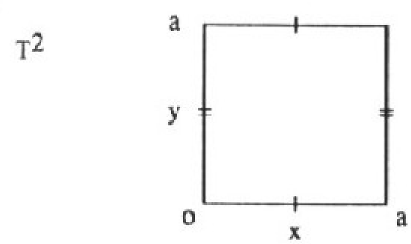
Chapter 2

Morse Theory on a 2-Torus

1. Morse Theory on a 2-Torus

To illustrate Witten's ideas about Morse Theory and supersymmetric quantum mechanics we will deal with one of the simplest compact manifolds, the 2-torus, which admits a metric of zero curvature. T^2 may be represented, in flat two-dimensional space, as a square with sides of length a, whose opposite edges are identified.



In two dimensions the possible different classes of p-forms are zero-forms $A(x, y)$, one-forms $B(x, y)dx$ and $C(x, y)dy$, and two-forms $D(x, y)dx \wedge dy$. To show in detail how various operators act on different p-forms the notation will be to represent wavefunctions as 4x1 column matrices

$$\Psi(x,y) = \begin{bmatrix} A(x,y) \\ B(x,y) \\ C(x,y) \\ D(x,y) \end{bmatrix}$$

and to represent operators such as the Hamiltonian or the supersymmetry operator as 4x4 matrices.

Taking the supersymmetry operator to be the sum of the exterior derivative and its adjoint $Q = d + \delta$, which in the matrix notation takes the form

$$Q = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 \end{pmatrix}$$

the Hamiltonian on T^2 is the flat space Laplacian

$$H = Q^2 = d\delta + \delta d$$

$$= [-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2}]\mathbb{I}$$

where $\mathbb{I}$ is the 4x4 unit matrix.

The Schrödinger Equation is

$$-\frac{\partial^2}{\partial x^2}\Psi(x,y) - \frac{\partial^2}{\partial y^2}\Psi(x,y) = E\Psi(x,y)$$

with boundary conditions

$$\Psi(0,y) = \Psi(a,y) \quad , \quad \frac{\partial\Psi(0,y)}{\partial x} = \frac{\partial\Psi(a,y)}{\partial x}$$

$$\Psi(x,0) = \Psi(x,a) \quad , \quad \frac{\partial\Psi(x,0)}{\partial y} = \frac{\partial\Psi(x,a)}{\partial y}$$

The zero energy solutions to this equation are the representatives of the cohomology of $\mathbf{T^2}$, these are the four constant p-forms

$$\Psi(x,y) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} , \begin{bmatrix}0\\1\\0\\0\end{bmatrix} , \begin{bmatrix}0\\0\\1\\0\end{bmatrix} , \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

thus the Betti Numbers $B_p(M)$, the dimensions of the independent harmonic p-forms, are $B_0(T^2) = 1$, $B_1(T^2) = 2$, $B_2(T^2) = 1$.

These four solutions are annihilated by the supersymmetry operator and so form singlets under supersymmetry. Higher energy solutions may be obtained by multiplying these four constant p-forms by the functions:

$$\chi^1_{n,m}(x,y) = sin[\tfrac{2n\pi x}{a}]sin[\tfrac{2m\pi y}{a}]$$

and

$$\chi^2_{n,m}(x,y) = cos[\tfrac{2n\pi x}{a}]cos[\tfrac{2m\pi y}{a}]$$

where n and m are integers. Along with their supersymmetry partners, this gives sixteen solutions each with energy $E = \frac{4\pi^2}{a^2}(n^2 + m^2)$. The supersymmetry partners can easily be obtained by applying the supersymmetry operator. For instance, acting on the zero-form solution $\chi^1_{n,m}(x,y)$, gives

$$Q\chi^1_{n,m}(x,y) = -\tfrac{2\pi}{a}[ncos[\tfrac{2n\pi x}{a}]sin[\tfrac{2m\pi y}{a}]dx + msin[\frac{2n\pi x}{a}]cos[\tfrac{2m\pi y}{a}]dy]$$

2.1 Morse Function (1)

A Morse Function on a torus is a smooth real-valued function defined on the torus i.e.

$$h : T^2 \to \mathbb{R}$$

so to write the function h in terms of Cartesian coordinates the function must have the same period as that of the torus itself. A suitable Morse Function for this example is

$$h(x, y) = sin^2[\tfrac{\pi x}{a}] + sin^2[\tfrac{\pi y}{a}]$$

The critical points of the Morse Function are the points at which the one-form $dh(x, y)$ vanishes, the maxima, minima and saddle points of the function

$$\frac{\partial h(x,y)}{\partial x} = \frac{2\pi}{a} sin[\tfrac{\pi x}{a}] cos[\tfrac{\pi x}{a}], \qquad \frac{\partial h(x,y)}{\partial y} = \frac{2\pi}{a} sin[\tfrac{\pi y}{a}] cos[\tfrac{\pi y}{a}]$$

$$\frac{\partial h(x,y)}{\partial x} = 0 \Rightarrow x = 0, \frac{a}{2} \quad , \qquad \frac{\partial h(x,y)}{\partial y} = 0 \Rightarrow y = 0, \frac{a}{2}$$

Thus the critical points of $h(x, y)$ are $(0, 0), (\frac{a}{2}, 0), (0, \frac{a}{2}), (\frac{a}{2}, \frac{a}{2})$.

The Morse Index of a critical point is the number of negative eigenvalues of the Hessian of $h(x, y)$ at the critical point.

$$\frac{\partial^2 h(x,y)}{\partial x^2} = \frac{2\pi^2}{a^2}[cos^2[\tfrac{\pi x}{a}] - sin^2[\tfrac{\pi x}{a}]] \quad , \qquad \frac{\partial^2 h(x,y)}{\partial y^2} = \frac{2\pi^2}{a^2}[cos^2[\tfrac{\pi y}{a}] - sin^2[\tfrac{\pi y}{a}]]$$

$$\frac{\partial^2 h(x,y)}{\partial x \partial y} = 0$$

giving the Hessian as

$$\begin{pmatrix} \frac{\partial^2 h(x,y)}{\partial x^2} & \frac{\partial^2 h(x,y)}{\partial x \partial y} \\ \frac{\partial^2 h(x,y)}{\partial x \partial y} & \frac{\partial^2 h(x,y)}{\partial y^2} \end{pmatrix} = \frac{2\pi^2}{a^2} \begin{pmatrix} cos(\frac{2\pi x}{a}) & 0 \\ 0 & cos(\frac{2\pi y}{a}) \end{pmatrix}$$

None of the eigenvalues at any of the critical points are zero so the Morse Function is non-degenerate.

A table of the critical points of h(x,y) on T² with their Morse Indices p and the critical value of h(x,y).

$critical point$	$Morse Index(p)$	$critical value of h(x, y)$
$(0, 0)$	0	0
$(\frac{a}{2}, 0)$	1	1
$(0, \frac{a}{2})$	1	1
$(\frac{a}{2}, \frac{a}{2})$	2	2

This shows that the Morse Numbers M_p, the number of critical points with Morse Index p, in this example are:

$$M_0 = 1, \qquad M_1 = 2, \qquad M_2 = 1,$$

so that the Morse Inequalities are saturated.

$$M_0 = B_0, \qquad M_1 = B_1, \qquad M_2 = B_2.$$

Introducing the Morse Funciton into the supersymmetry algebra by conjugating the exterior derivative with its exponential

$$d \rightarrow d_t = e^{-th} d e^{th}, \;\; \delta \rightarrow \delta_t = e^{th} \delta e^{-th}$$

which in terms of the matrix notation gives

$$d_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial x} + t\frac{\partial h(x,y)}{\partial x} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} + t\frac{\partial h(x,y)}{\partial y} & 0 & 0 & 0 \\ 0 & -\frac{\partial}{\partial y} - t\frac{\partial h(x,y)}{\partial y} & \frac{\partial}{\partial x} + t\frac{\partial h(x,y)}{\partial x} & 0 \end{pmatrix}$$

$$\delta_t = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + t\frac{\partial h(x,y)}{\partial x} & -\frac{\partial}{\partial y} + t\frac{\partial h(x,y)}{\partial y} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial y} - t\frac{\partial h(x,y)}{\partial y} \\ 0 & 0 & 0 & -\frac{\partial}{\partial x} + t\frac{\partial h(x,y)}{\partial x} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The supersymmetry operator generalizes to $Q_t = d_t + \delta_t$, and its square is the Hamiltonian H_t.

$$H_t = Q_t^2 = d_t\delta_t + \delta_t d_t =$$

$$\begin{pmatrix} \Box - t(\frac{\partial^2 h(x,y)}{\partial x^2} + \frac{\partial^2 h(x,y)}{\partial y^2}) & 0 & 0 & 0 \\ 0 & \Box - t(-\frac{\partial^2 h(x,y)}{\partial x^2} + \frac{\partial^2 h(x,y)}{\partial y^2}) & 0 & 0 \\ 0 & 0 & \Box - t(\frac{\partial^2 h(x,y)}{\partial x^2} - \frac{\partial^2 h(x,y)}{\partial y^2}) & 0 \\ 0 & 0 & 0 & \Box + t(\frac{\partial^2 h(x,y)}{\partial x^2} + \frac{\partial^2 h(x,y)}{\partial y^2}) \end{pmatrix}$$

where $\Box = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + t^2([\frac{\partial h}{\partial x}]^2 + [\frac{\partial h}{\partial y}]^2)$.

In terms of the Morse Function

$$h(x,y) = sin^2(\frac{\pi x}{a}) + sin^2(\frac{\pi y}{a})$$

the Hamiltonian takes the form

$$\begin{pmatrix} \Box - \frac{2\pi^2 t}{a^2}(cos(\frac{2\pi x}{a}) + cos(\frac{2\pi y}{a})) & 0 & 0 & 0 \\ 0 & \Box + \frac{2\pi^2 t}{a^2}(cos(\frac{2\pi x}{a}) - cos(\frac{2\pi y}{a})) & 0 & 0 \\ 0 & 0 & \Box - \frac{2\pi^2 t}{a^2}(cos(\frac{2\pi x}{a}) - cos(\frac{2\pi y}{a})) & 0 \\ 0 & 0 & 0 & \Box + \frac{2\pi^2 t}{a^2}(cos(\frac{2\pi x}{a}) + cos(\frac{2\pi y}{a})) \end{pmatrix}$$

where $\Box = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{\pi^2 t^2}{a^2}(sin^2[\frac{2\pi x}{a}] + sin^2[\frac{2\pi y}{a}])$.

The Schrödinger Equation is no longer exactly soluble for excited states, but exact zero energy solutions can still be found. The zero energy solutions are:

$$exp[-t(sin^2[\tfrac{\pi x}{a}] + sin^2[\tfrac{\pi y}{a}])] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$exp[-t(cos^2[\tfrac{\pi x}{a}] + sin^2[\tfrac{\pi y}{a}])] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$exp[-t(sin^2[\tfrac{\pi x}{a}] + cos^2[\tfrac{\pi y}{a}])] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$exp[-t(cos^2[\tfrac{\pi x}{a}] + cos^2[\tfrac{\pi y}{a}])] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

corresponding to the Betti Numbers of the Torus being $B_0(T^2) = 1$, $B_1(T^2) = 2$, $B_2(T^2) = 1$.

2.1.1 The large t limit

For large t the Hamiltonian is large everywhere except near the critical points of h(x,y), where $\dfrac{\partial h(x,y)}{\partial x}$ and $\dfrac{\partial h(x,y)}{\partial y}$ simultaneously vanish, so the low energy solutions of the Schrödinger Equation become concentrated around these points. Near the critical point (0,0), taking x and y to be small and neglecting terms of $O(x^2), O(y^2)$, the operators d_t and δ^2 take the approximate forms $\tilde{d}_t$, $\tilde{\delta}_t$, where

$$\tilde{d}_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial x} + 2t\frac{\pi^2}{a^2}x & 0 & 0 & 0 \\ \frac{\partial}{\partial y} + 2t\frac{\pi^2}{a^2}y & 0 & 0 & 0 \\ 0 & -\frac{\partial}{\partial y} - 2t\frac{\pi^2}{a^2}y & \frac{\partial}{\partial x} + 2t\frac{\pi^2}{a^2}x & 0 \end{pmatrix}$$

$$\tilde{\delta}_t = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + 2t\frac{\pi^2}{a^2}x & -\frac{\partial}{\partial y} + 2t\frac{\pi^2}{a^2}y & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial y} - 2t\frac{\pi^2}{a^2}y \\ 0 & 0 & 0 & -\frac{\partial}{\partial x} + 2t\frac{\pi^2}{a^2}x \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so the approximate Hamiltonian is that of a supersymmetric two-dimensional harmonic oscillator around the critical point.

$$\tilde{H}_t = \tilde{d}_t\tilde{\delta}_t + \tilde{\delta}_t\tilde{d}_t = \begin{pmatrix} \tilde{\Box} - 4\frac{\pi^2}{a^2}t & 0 & 0 & 0 \\ 0 & \tilde{\Box} & 0 & 0 \\ 0 & 0 & \tilde{\Box} & 0 \\ 0 & 0 & 0 & \tilde{\Box} + 4\frac{\pi^2}{a^2}t \end{pmatrix}$$

where $\tilde{\Box} = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 4\frac{\pi^4}{a^4}t^2(x^2 + y^2)$

The supersymmetric harmonic oscillator has one zero energy solution, which in this case is the zero-form:

$$|0,0>= \tilde{\Psi}_{0,0}(x,y) = exp[-\frac{\pi^2}{a^2}t(x^2+y^2)] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where the otherwise non-zero zero point energy is cancelled in the Hamiltonian by the term $-4\frac{\pi^4}{a^2}t$. Higher energy solutions can be found by acting on this ground state with the harmonic oscillator ladder operators. The raising operators are:

$$X_+ = \frac{\partial}{\partial x} - 2\frac{\pi^2}{a^2}tx \quad , \qquad Y_+ = \frac{\partial}{\partial y} - 2\frac{\pi^2}{a^2}ty$$

each of which acting on a solution of energy E gives a solution of energy $E + 4\frac{\pi^2}{a^2}t$ and the lowering operators are:

$$X_- = \frac{\partial}{\partial x} + 2\frac{\pi^2}{a^2}tx \quad , \qquad Y_- = \frac{\partial}{\partial y} + 2\frac{\pi^2}{a^2}ty \; .$$

The complete zero-form solutions are:

$$\tilde{\Psi}_{n,m}(x,y) = (X_+)^n (Y_+)^m \tilde{\Psi}_{0,0}(x,y) = X_+^n Y_+^m |0,0>$$

with energy $E = (n+m)\dfrac{4\pi^2}{a^2}t$.

The spectrum for one-form solutions is exactly the same but with the energy of each state raised by $\dfrac{4\pi^2}{a^2}t$,

$$\begin{aligned}
&\tilde{\Psi}_{n,m}(x,y)dx = (X_+)^n (Y_+)^m \tilde{\Psi}_{0,0}(x,y)dx \\
&\tilde{\Psi}_{n,m}(x,y)dy = (X_+)^n (Y_+)^m \tilde{\Psi}_{0,0}(x,y)dy \\
&E = (n+m+1)\frac{4\pi^2}{a^2}t
\end{aligned}$$

and the two-form solutions have energy raised by $\dfrac{8\pi^2}{a^2}t$

$$\begin{aligned}
&\tilde{\Psi}_{n,m}(x,y)dx \wedge dy = (X_+)^n (Y_+)^m \tilde{\Psi}_{0,0}(x,y)dx \wedge dy \\
&E = (n+m+2)\frac{4\pi^2}{a^2}t
\end{aligned}$$

The structure of the supersymmetry doublets can be seen by writing out the supersymmetry operator explicitly:

$$\tilde{Q}_t = \tilde{d}_t + \tilde{\delta}_t = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + 2t\frac{\pi^2}{a^2}x & -\frac{\partial}{\partial y} + 2t\frac{\pi^2}{a^2}y & 0 \\ \frac{\partial}{\partial x} + 2t\frac{\pi^2}{a^2}x & 0 & 0 & \frac{\partial}{\partial y} - 2t\frac{\pi^2}{a^2}y \\ \frac{\partial}{\partial y} + 2t\frac{\pi^2}{a^2}y & 0 & 0 & -\frac{\partial}{\partial x} + 2t\frac{\pi^2}{a^2}x \\ 0 & -\frac{\partial}{\partial y} - 2t\frac{\pi^2}{a^2}y & \frac{\partial}{\partial x} + 2t\frac{\pi^2}{a^2}x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -X_+ & -Y_+ & 0 \\ X_- & 0 & 0 & Y_+ \\ Y_- & 0 & 0 & -X_+ \\ 0 & Y_- & X_- & 0 \end{pmatrix}$$

so acting on the zero-form $\tilde{\Psi}_{n,m}(x,y)\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ with $\tilde{Q}_t$ will give its superpartner $\tilde{\Psi}_{n-1,m}(x,y)\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+\tilde{\Psi}_{n,m-1}(x,y)\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$. Similarly for the two-form solutions, $\tilde{Q}_t$ pairs $\tilde{\Psi}_{n,m}(x,y)\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ with the linear combination of one-forms $\tilde{\Psi}_{n,m+1}(x,y)\begin{bmatrix}0\\1\\0\\0\end{bmatrix}-\tilde{\Psi}_{n+1,m}(x,y)\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$.

Around the other critical points the results are completely analogous, but with the spectrum of the different kinds of p-forms interchanged.

Near the critical point $(\frac{a}{2},0)$ which has Morse Index 1:

$$\tilde{H}=\begin{pmatrix}\tilde{\Box} & 0 & 0 & 0\\ 0 & \tilde{\Box}-4\frac{\pi^2}{a^2}t & 0 & 0\\ 0 & 0 & \tilde{\Box}+4\frac{\pi^2}{a^2}t & 0\\ 0 & 0 & 0 & \tilde{\Box}\end{pmatrix}$$

where $\tilde{\Box}=-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+4\frac{\pi^4}{a^4}t^2([x-\frac{a}{2}]^2+y^2)$.

The zero energy solution is the one-form $|\dfrac{a}{2},0>$:

$$|\tfrac{a}{2},0>=\tilde{\Psi}_{0,0}(x-\tfrac{a}{2},y)dx=exp[-\tfrac{\pi^2}{a^2}t([x-\tfrac{a}{2}]^2+y^2)]\begin{bmatrix}0\\1\\0\\0\end{bmatrix}.$$

Near the critical point $(0,\dfrac{a}{2})$ which has Morse Index 1, the zero energy solution $|0,\dfrac{a}{2}>$ is:

$$|0,\tfrac{a}{2}>=\tilde{\Psi}_{0,0}(x,y-\tfrac{a}{2})dy=exp[-\tfrac{\pi^2}{a^2}t(x^2+[y-\tfrac{a}{2}]^2)]\begin{bmatrix}0\\0\\1\\0\end{bmatrix}.$$

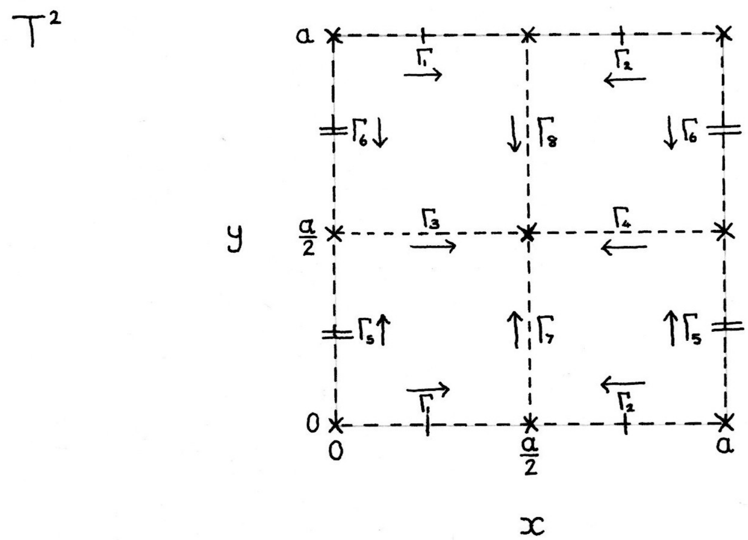
Near the critical point $(\dfrac{a}{2},\dfrac{a}{2})$ which has Morse Index 2, the zero energy solution is the two-form:

$$|\frac{a}{2},\frac{a}{2}>=\tilde{\Psi}_{0,0}(x-\tfrac{a}{2},y-\tfrac{a}{2})dx\wedge dy=exp[-\tfrac{\pi^2}{a^2}t([x-\tfrac{a}{2}]^2+[y-\tfrac{a}{2}]^2)]\begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

At each critical point of Morse Index p, the zero energy solution is a p-form.

2.1.2 Tunnelling (1)

Because the Morse Inequalities are saturated in this example, these approximate zero energy solutions must correspond to the zero energy solutions of the exact Schrödinger Equation. Any tunnelling between critical points which might have removed the degeneracy between these energy levels cancels because there are two paths in opposite directions between any two critical points differing in Morse Index by one.



The diagram shows the critical points x, of the Morse Function, with paths of steepest ascent Γ_i between critical points and arrows showing the direction of the path. The coboundary operator $\triangle$ and its adjoint annihilate all of the zero energy states localized at the critical points. For example, acting on the one-form state $|0, \frac{a}{2}>$, localized at the point $(0, \frac{a}{2})$

$$\begin{aligned}\triangle|0, \frac{a}{2} > &= n[(0, \frac{a}{2}), (\frac{a}{2}, \frac{a}{2})]|\frac{a}{2}, \frac{a}{2} > \\ &= \sum_{\Gamma_i} (-1)^{n\Gamma_i}|\frac{a}{2}, \frac{a}{2} > \\ &= (-1)^{n\Gamma_3}|\frac{a}{2}, \frac{a}{2} > + (-1)^{n\Gamma_4}|\frac{a}{2}, \frac{a}{2} > \\ &= |\frac{a}{2}, \frac{a}{2} > - |\frac{a}{2}, \frac{a}{2} > \\ &= 0\end{aligned}$$

See the introductory section 1.2 for the definition of the operator $\triangle$.

The next example of a Morse Function will be in the more interesting case where the Morse Inequalities are not saturated.

2.2 Morse Function (2)

A less trivial example of a Morse Function, where the Morse Inequalities are not saturated, may be found by halving the period in the x direction of the Morse Function in the previous example. This gives the new Morse Function:

$$h(x,y) = sin^2[\tfrac{2\pi x}{a}] + sin^2[\tfrac{\pi y}{a}]$$

Differentiating to find the critical points gives:

$$\frac{\partial h(x,y)}{\partial x} = \frac{4\pi}{a} sin[\tfrac{2\pi x}{a}]cos[\tfrac{2\pi x}{a}] \qquad , \qquad \frac{\partial h(x,y)}{\partial y} = \frac{2\pi}{a} sin[\tfrac{\pi y}{a}]cos[\tfrac{\pi y}{a}]$$

$$\frac{\partial h(x,y)}{\partial x} = 0 \Rightarrow x = 0, \frac{a}{4}, \frac{a}{2}, \frac{3a}{4} \qquad , \qquad \frac{\partial h(x,y)}{\partial y} = 0 \Rightarrow y = 0, \frac{a}{2}.$$

Therefore h(x,y) has eight critical points. Differentiating again gives the Hessian as:

$$\frac{2\pi^2}{a^2}\begin{pmatrix} 4cos(\frac{4\pi x}{a}) & 0 \\ 0 & cos(\frac{2\pi y}{a}) \end{pmatrix}.$$

A table of the critical points, their Morse Index and the critical value of h(x,y).

$critical point$	$Morse Index(p)$	$critical value of h(x,y)$
$(0,0)$	0	0
$(0,\frac{a}{2})$	1	1
$(\frac{a}{4},0)$	1	1
$(\frac{a}{4},\frac{a}{2})$	2	2
$(\frac{a}{2},0)$	0	0
$(\frac{a}{2},\frac{a}{2})$	1	1
$(\frac{3a}{4},0)$	1	1
$(\frac{3a}{4},\frac{a}{2})$	2	2

Thus the Morse Numbers are:

$M_0 = 2, \qquad M_1 = 4, \qquad M_2 = 2,$

so the Morse Inequalities are not saturated. The Morse Inequalities state:

$$(M_0 = 2) \geq (B_0 = 1)$$
$$(M_1 = 4) - (M_0 = 2) \geq (B_1 = 2) - (B_0 = 1)$$
$$(M_2 = 2) - (M_1 = 4) + (M_0 = 2) = (B_2 = 1) - (B_1 = 2) + (B_0 = 1) = \chi(T^2) = 0$$

Introducing the Morse Function into the supersymmetry algebra leads to a Hamiltonian that is almost identical to the previous example:

$H_t =$

$$\begin{pmatrix} \Box - \frac{2\pi^2 t}{a^2}(4cos(\frac{4\pi x}{a}) + cos(\frac{2\pi y}{a})) & 0 & 0 & 0 \\ 0 & \Box + \frac{2\pi^2 t}{a^2}(4cos(\frac{4\pi x}{a}) - cos(\frac{2\pi y}{a})) & 0 & 0 \\ 0 & 0 & \Box - \frac{2\pi^2 t}{a^2}(4cos(\frac{4\pi x}{a}) - cos(\frac{2\pi y}{a})) & 0 \\ 0 & 0 & 0 & \Box + \frac{2\pi^2 t}{a^2}(4cos(4\frac{2\pi x}{a}) + cos(\frac{2\pi y}{a})) \end{pmatrix}$$

where $\Box = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{\pi^2 t^2}{a^2}(4sin^2[\frac{4\pi x}{a}] + sin^2[\frac{2\pi y}{a}])$.

The Schrödinger Equation has four zero energy solutions, a zero-form, two one-forms and a two-form, corresponding to the cohomology of the torus. These solutions are:

$$exp[-t(sin^2[\frac{2\pi x}{a}] + sin^2[\frac{\pi y}{a}])] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$exp[-t(cos^2[\frac{2\pi x}{a}] + sin^2[\frac{\pi y}{a}])] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$exp[-t(sin^2[\frac{2\pi x}{a}] + cos^2[\frac{\pi y}{a}])] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$exp[-t(cos^2[\frac{2\pi x}{a}] + cos^2[\frac{\pi y}{a}])] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

For large t the Hamiltonian approximates to a harmonic oscillator at each of the critical points, and so there is one p-form approximate zero-energy solution around each critical point of Morse Index p. At the critical point (x_c, y_c) this solution takes the form:

$$exp[-t\frac{\pi^2}{a^2}[4(x - x_c)^2 + (y - y_c)^2]]$$

multiplied by the p-form corresponding to the critical point. In the case of the critical points of Morse Index one, the points $(\frac{a}{4}, 0), (\frac{3a}{4}, 0)$ correspond to the one-form dx and the points $(0, \frac{a}{2}), (\frac{a}{2}, \frac{a}{2})$ correspond to the one-form dy.

In this example there are twice as many approximate zero energy states as exact zero energy solutions. A more exact calculation must remove this spurious degeneracy. Perturbation theory, however, will not lift the energy of any of the approximate zero energy states. At each critical point the harmonic oscillator solutions form a complete set of orthogonal functions, but solutions at different critical points will not be orthogonal, so perturbation theory must be done separately at each critical point and degenerate perturbation theory will not be used even though the zero energy approximate solutions are degenerate. If in perturbation theory any of these approximate zero energy solutions gained a contribution to their energy, then by the symmetry of the problem under a translation on a torus, all the approximate zero energy solutions must gain an energy. This cannot occur because there are in fact zero energy solutions. An alternative way of looking at this is that to whatever order the Morse Function $(x - x_c)$ is expanded in $\tilde{h}(x - x_c, y - y_c)$ and $(y - y_c)$, around the critical point, in the Schrödinger Equation, there is always one zero energy solution which is $exp[-t\tilde{h}(x - x_c, y - y_c)]$ multiplied by the p-form corresponding to the critical point. This zero energy solution will always be normalizable as it is defined on a compact manifold. For instance, expanding the Morse Function to third order around the critical point (0,0):

$$d_t = d + \frac{\partial h(x,y)}{\partial x} dx \wedge + \frac{\partial h(x,y)}{\partial y} dy \wedge$$
$$\tilde{d_t} = d + [8x - \tfrac{64}{3}x^3]dx \wedge + [2y - \tfrac{16}{3}y^3]dy \wedge \, .$$

The approximated operator $\tilde{d_t}$ will annihilate the function:

$$exp[-t\tilde{h}(x,y)] = exp[-t(4x^2 - \tfrac{16}{3}x^4 + y^2 - \tfrac{4}{3}y^4)]$$

which is normalizable because x and y are only defined within the range $0 < x, y \leq a$.
Excited states, however, will not in general have zero contributions in perturbation theory and their energy will form an asymptotic series in powers of $\frac{1}{t}$. For example the first excited state:

$$\tilde{\Psi}_{0,1}(x,y) \approx y exp[-t(4x^2 + y^2)]$$

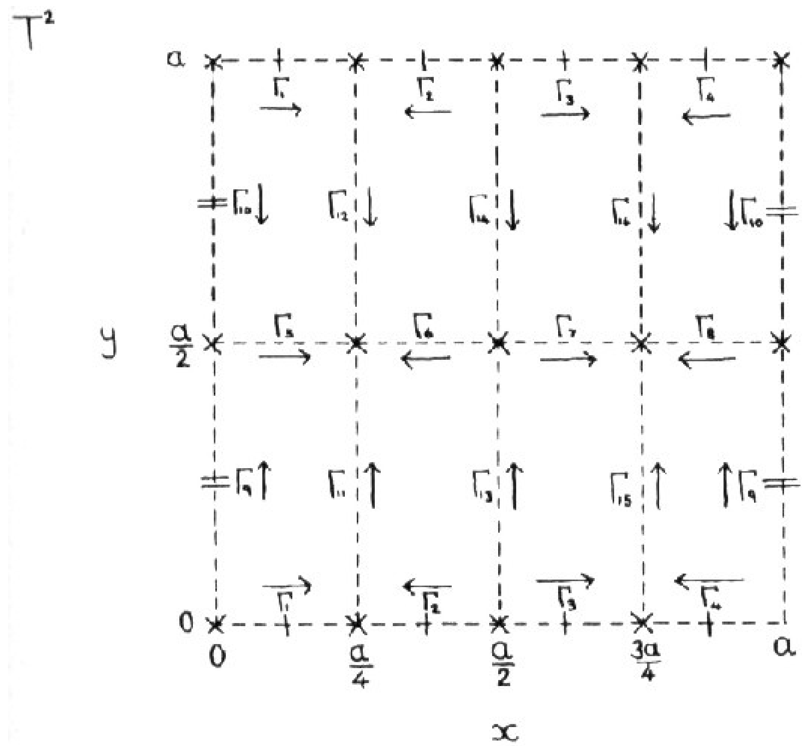
after first order perturbation theory is found to have energy $E = 2t - 2 + O(\frac{1}{t})$. To all orders of perturbation theory this state will be degenerate with

$$\tilde{\Psi}_{0,1}(x - \frac{a}{2}, y) \approx y exp[-t(4(x - \frac{a}{2})^2 + y^2)]$$

as all the integrals involved will be equal up to a translation in x, so as with the zero energy states the degeneracy may only be split by non-perturbative or tunnelling effects.

2.2.1 Tunnelling (2)

The Morse Inequalities are not saturated in this example, so tunnelling effects may produce an improved bound on the Betti Numbers.



The diagram shows the critical points x, and the paths of steepest ascent between them marked by the dashed lines and labelled Γ_i . The arrows show the direction of the path i.e. from a point of Morse Index p to a critical point of Morse Index (p+1).

All contributions to the operation of the coboundary operator $\triangle$, from paths of steepest descent in the y direction cancel due to there being two paths in opposite directions in each case. This is not true of paths in the x direction. For example, acting with the coboundary operator $\triangle$ on the zero-form state $|0, 0>$ gives:

$$
\begin{aligned}
\triangle|0,0> &= \sum_b n((0,0),B)|b> \\
&= n((0,0),(\frac{a}{4},0))|\frac{a}{4},0> + n((0,0),(\frac{3a}{4},0))|\frac{3a}{4},0> + n((0,0),(0,\frac{a}{2}))|0,\frac{a}{2}> \\
&= (-1)^{n\Gamma_1}|\frac{a}{4},0> + (-1)^{n\Gamma_4}|\frac{3a}{4},0> + (-1)^{n\Gamma_9}|0,\frac{a}{2}> + (-1)^{n\Gamma_{10}}|0,\frac{a}{2}> \\
&= |\frac{a}{4},0> - |\frac{3a}{4},0> .
\end{aligned}
$$

Acting on the other approximate zero energy zero-form state $|\frac{a}{2}, 0>$, $\triangle$ gives the one-form state:

$$
\triangle|\frac{a}{2},0> = |\frac{3a}{4},0> - |\frac{a}{4},0>
$$

so that the zero-form state annihilated by this operator, the state which still has zero energy in the W.K.B. approximation, is the symmetric combination

$$|0,0> +|\frac{a}{2},0> .$$

The anti-symmetric zero-form

$$|0,0> -|\frac{a}{2},0>$$

being paired by the operation of $\triangle$ to form a supersymmetry doublet with the one-form,

$$|\frac{a}{4},0> -|\frac{3a}{4},0> .$$

Similarly the coboundary operator $\triangle$ annihilates the symmetric one-form states,

$$|\frac{a}{4},0> +|\frac{3a}{4},0> \qquad , \qquad |0,\frac{a}{2}> +|\frac{a}{2},\frac{a}{2}>$$

and pairs the anti-symmetric one-form

$$|0,\frac{a}{2}> -|\frac{a}{2},\frac{a}{2}>$$

and the anti-symmetric two-form state

$$|\frac{a}{4},\frac{a}{2}> -|\frac{3a}{4},\frac{a}{2}> .$$

The symmetric two-form state

$$|\frac{a}{4},\frac{a}{2}> +|\frac{3a}{4},\frac{a}{2}>$$

being a two-form, is automatically annihilated by $\triangle$ which will map p-forms to (p-1)-forms with signs dependent on the directions of the paths between the critical points.

In this example, the number of zero energy p-form solutions in the W.K.B. approximation are:

$$Y_0 = 1, \qquad Y_1 = 2, \qquad Y_2 = 1.$$

These numbers equal the Betti Numbers of the torus T^2.