Chapter 3

Fixed Point Theorems

Appendix 3(i)

Topological Definitions and Results [5], [9]

Much of this thesis is concerned with differential topology on compact Riemannian manifolds. This appendix contains definitions of certain of the operators which are used. The objects on which these operators act are differential forms, ω_p, which are the totally anti-symmetric covariant tensor fields

$$\omega_p = f(x)_{i_1 i_2 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

The wedge product being the anti-symmetric tensor product.

The exterior derivative maps p-forms to (p+1)-forms according to the rule:

$$d\omega_p = \frac{\partial}{\partial x_{i_{p+1}}} f(x)_{i_1 i_2 \cdots i_p} dx^{i_{p+1}} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

where the new differential line element is introduced before any previously existing wedge products. The exterior derivative gives zero when applied twice due to the symmetry under the interchange of the two derivatives and the anti-symmetry of the wedge product.

The interior product of a vector $\mathbf{k} = k^i \frac{\partial}{\partial x_i}$ with a p-form is defined as:

$$i_{\mathbf{k}} \omega_p = k^i f_{i_1 i_2 \cdots i_p} dx^{i_2} \wedge \cdots \wedge dx^{i_p}$$

mapping p-forms to (p-1)-forms. This is also a nilpotent operation.

There is a natural correspondence between the space of p-forms and the space of (n-p)-forms. This motivates the introduction of the Hodge Star Operation which transforms p-forms into (n-p)-forms. The Hodge Star is defined as:

$$^*(dx^{i_1} \wedge \cdots \wedge dx^{i_p}) = \frac{|g|^2}{(n-p)!} \epsilon^{i_1 \cdots i_p}_{i_{p+1} \cdots i_n} dx^{i_{p+1}} \wedge \cdots \wedge dx^{i_n},$$

where g is the determinant of the metric tensor. Performing this operation twice in succession produces the result:

$$^{**}\omega_p = (-1)^{p(n-p)} \omega_p$$

where n is the dimension of the manifold.

The adjoint of an operator is defined via the inner product

$$(\alpha_p, \beta_p) = \int_M \alpha_p \wedge {}^*\beta_p \,.$$

The definition of the adjoint of the exterior derivative is

$$(d\alpha_{p-1}, \beta_p) = (\alpha_{p-1}, \delta\beta_p)\,.$$

An explicit formula for δ may be found by equating the integral of a total derivative over a compact manifold without boundary to zero using Stokes' Theorem.

$$\int d[\alpha_{p-1} \wedge {}^*\beta_p] = 0 = \int [d\alpha_{p-1}] \wedge {}^*\beta_p + (-1)^{p-1} \int \alpha_{p-1} \wedge d[{}^*\beta_p] \, .$$

Inserting the square of the Hodge Star in the second integral gives

$$(d\alpha_{p-1}, \beta_p) = -(-1)^{(p-1)}(-1)^{(n-p+1)(p-1)}(\alpha_{p-1}, {}^*d^*\beta_p)$$

resulting in the formula

$$\delta = (-1)^{(np+n+1)*}d^* \, .$$

A p-form which can be written globally as the exterior derivative of a (p-1)-form, $\omega_p = d\alpha_{p-1}$, is called an exact p-form. A p-form which can be written globally as $\omega_p = \delta\alpha_{p+1}$ is called co-exact. Hodge's Theorem states that on a compact manifold without boundary any p-form can be uniquely decomposed into the sum of an exact, a co-exact and a harmonic piece

$$\omega_p = d\alpha_{p-1} + \delta\beta_{p+1} + \gamma_p$$

where harmonic means that the p-form is annihilated by both the exterior derivative and its adjoint.

The adjoint of the interior product with a vector **k** is the exterior product with with $\tilde{\mathbf{k}}$, the one-form dual to **k**. This is defined in a completely analogous way

$$(i_{\mathbf{k}}\alpha_{p+1}, \beta_p) = (\alpha_{p+1}, e_{\tilde{\mathbf{k}}}\beta_p) \, .$$

The explicit form of $e_{\tilde{\mathbf{k}}}$ may be found by using the fact that $\alpha_{p+1} \wedge *\beta_p$ is an (n+1)-form and therefore identically zero.

$$\int i_{\mathbf{k}}[\alpha_{p+1} \wedge {}^*\beta_p] = 0 = \int [i_{\mathbf{k}}\alpha_{p+1}] \wedge {}^*\beta_p + (-1)^{p+1} \int \alpha_{p+1} \wedge i_{\mathbf{k}}[{}^*\beta_p] \, .$$

Inserting the square of the Hodge Star, as previously, leads to the formula for the exterior product

$$e_{\tilde{\mathbf{k}}} = (-1)^{(np+n+1)*}i_{\mathbf{k}}{}^* \qquad .$$

The dual one-form $\tilde{\mathbf{k}} = k_j dx^j$ may be obtained from the vector $\mathbf{k} = k^i \frac{\partial}{\partial x_i}$ using the metric tensor $k_j = g_{ij}k^i$. The function $K^2 = i_{\mathbf{k}}\tilde{\mathbf{k}}$ is therefore

$$K^2 = k_j k^j = g_{ij}k^i k^j \, .$$

A positive definite operator, the Laplacian $\triangle_S$, may be defined in terms of an operator, say d_S, and its adjoint δ_S.

$$\triangle_S = d_S\delta_S + \delta_S d_S \, .$$

The fact that this operator is positive definite may be demonstrated by consideration of the inner product of the p-forms ω_p and $\triangle_S\omega_p$.

$$\begin{aligned}(\omega_p, \triangle_S\omega_p) &= (\omega_p, d_S\delta_S\omega_p) + (\omega_p, \delta_S d_S\omega_p) \\ &= (\delta_S\omega_p, \delta_S\omega_p) + (d_S\omega_p, d_S\omega_p) \\ &\geq 0\end{aligned}$$

Any eigenvalue of the Laplacian must therefore be greater than or equal to zero because if ω_p is a normalized eigenform with eigenvalue E, then $(\omega_p, \triangle_S\omega_p) = E$.

3. Fixed Point Theorems

As with the Morse Inequalities, Witten uses an index theorem method to prove the Lefschetz Fixed Point Theorems, but by employing a slightly more complicated supersymmetric system. These theorems equate the Euler Characteristic and Hirzebruch Signature of a manifold to those of any submanifold which is invariant under the action of a vector. In what follows only the even-dimensional case will be considered, as the Fixed Point Theorems are trivial for odd-dimensional manifolds, see Appendix 3(ii).

On a compact, oriented Riemannian manifold M, which admits the action of a one-parameter group of isometries generated by a Killing Vector **k**, the exterior derivative may be generalized to

$$d_s = d + s i_{\mathbf{k}}$$

where s is an arbitrary parameter, and $i_{\mathbf{k}}$ is the interior product with the Killing Vector. The square of this operator is the Lie Derivative along the Killing Vector.

$$d_s{}^2 = s(d i_{\mathbf{k}} + i_{\mathbf{k}} d) = s\mathcal{L}_{\mathbf{k}} \, .$$

In the even-dimensional case the adjoint of d_s is the operator δ_s, where

$$\delta_s = -{}^*d^* - s^* i_{\mathbf{k}}^* = \delta + s e_{\tilde{\mathbf{k}}} \, .$$

The operator $e_{\tilde{\mathbf{k}}} = -{}^* i_{\mathbf{k}}^*$ is the exterior product with $\tilde{\mathbf{k}}$, the one-form dual to the vector **k**, as defined on the third page of Appendix 3(i). The square of the Hodge Star operation is

$$^{**}\omega_p = (-1)^p \omega_p$$

where ω_p is a p-form. This leads to the result that the square of the adjoint operator δ_s is minus the Lie Derivative along the Killing Vector.

$$\begin{aligned}
\delta_s{}^2 &= -s({}^*d^{**} i_{\mathbf{k}}^* + {}^* i_{\mathbf{k}}^{**} d^*) \\
&= -s((-1)^{n-p-1*} d i_{\mathbf{k}}^* + (-1)^{n-p+1*} i_{\mathbf{k}} d^*) \\
&= (-1)^{p+1} s^* (d i_{\mathbf{k}} + i_{\mathbf{k}} d)^* \\
&= (-1)^{p+1} s^* \mathcal{L}_{\mathbf{k}}{}^* \\
&= (-1)^{2p+1} s \mathcal{L}_{\mathbf{k}} \\
&= -s\mathcal{L}_{\mathbf{k}} \, .
\end{aligned}$$

This result, that $\delta_s{}^2 = -d_s{}^2$, only occurs due to **k** being a Killing Vector. Because **k** generates an isometry the Lie Derivative acting on the metric gives zero, $\mathcal{L}_{\mathbf{k}} g = 0$, and so the Hodge Star operation commutes with $\mathcal{L}_{\mathbf{k}}$.

3.1 The Lefschetz Fixed Point Theorem

Defining the supersymmetry operator in terms of the generalized exterior derivative and its adjoint

$$D_s = d_s + \delta_s$$

gives the Hamiltonian as

$$H_s = D_s^2 = d_s\delta_s + \delta_s d_s + d_s^2 + \delta_s^2 = d_s\delta_s + \delta_s d_s \ .$$

The operator δ_s maps p-forms to (p+1)-forms and (p-1)-forms, so the supersymmetry operator maps even-forms to odd-forms and odd-forms to even-forms. Acting on the exterior algebra decomposed into $\Lambda^{even}(M)$ and $\Lambda^{odd}(M)$, the index of the supersymmetry operator is the number of even-form eigensolutions of H_s minus the number of odd-form eigensolutions. The Lefschetz Fixed Point Theorem may be proved by arguing that this index is independent of s, and then studying the large s spectrum of the Hamiltonian equating the value of the index when s = 0 and s tends to infinity. The reason that the index does not vary with s derives from the fact that all non-zero energy solutions of the Schrödinger Equation form quadruplets, $(\Psi(x), d_s\Psi(x), \delta_s\Psi(x), d_s\delta_s\Psi(x))$, two even-forms and two odd-forms, whose contribution to the index is therefore zero.

The degeneracy of these solutions may easily be demonstrated. Taking $\Psi(x)$ to be a solution of the Schrödinger Equation with energy E,

$$(d_s\delta_s + \delta_s d_s)\Psi(x) = E\Psi(x)$$

and acting with the operator d_s, gives

$$d_s(d_s\delta_s + \delta_s d_s)\Psi(x) = E d_s\Psi(x)$$

$d_s^2 = \mathcal{L}_{\mathbf{k}} = -\delta_s^2$ implies $[\mathcal{L}_{\mathbf{k}}, \delta_s] = [d_s^2, \delta_s] = 0$.

The fact that δ_s commutes with d_s^2 gives

$$(\delta_s d_s^2 + d_s\delta_s d_s)\Psi(x) = E d_s\Psi(x)$$
$$(\delta_s d_s + d_s\delta_s) d_s\Psi(x) = E d_s\Psi(x)$$

so $d_s\Psi(x)$ is also a solution of the Schrödinger Equation with energy E. Similarly δ_s and $d_s\delta_s$ commute with the Hamiltonian to give new solutions $\delta_s\Psi(x)$ and $d_s\delta_s\Psi(x)$, with energy E. These are all the solutions which are degenerate due to supersymmetry because $d_s^2\Psi(x) = \mathcal{L}_{\mathbf{k}}\Psi(x)$. $\mathcal{L}_{\mathbf{k}}$ commutes with the Hamiltonian, so $\Psi(x)$ may be taken to be a simultaneous eigensolution of both of these operators. $\delta_s d_s\Psi(x) = E\Psi(x) - d_s\delta_s\Psi(x)$, so this also does not give a new solution.

The Hodge Star * also commutes with the Hamiltonian thus doubling the degeneracy of the excited states to eightfold.

The zero energy solutions are supersymmetry singlets, annihilated by d_s and δ_s . The index of the supersymmetry operator acting on $\Lambda^{even}(M)$ and $\Lambda^{odd}(M)$ is therefore the number of zero energy even-form solutions minus the number of zero energy odd-form solutions. As the parameter s is varied continuously the index cannot change. For some value of s a solution with zero energy may obtain a positive energy or the energy of one that was positive may fall to zero, but any solutions whose energy, as a function of s, behaves in this way must be a member of a degenerate quadruplet, so the index will not change.

When s = 0 the supersymmetry operator is $Q = d + \delta$, and the index is the index of the De Rham complex which is equal to the Euler Characteristic of the manifold $\chi(M)$. The value of the index for large s may be found by a closer examination of the Hamiltonian and turns out to equal the Euler Characteristic $\chi(N)$, of the fixed submanifold N. The fixed submanifold consists of the set of points which are mapped into themselves by the action of the isometry, i.e. the points at which all the components of the Killing Vector vanish.

Writing out the expression for the Hamiltonian more explicitly

$$\begin{aligned} H_s &= d_s\delta_s + \delta_s d_s \\ &= (d + si_{\mathbf{k}})(\delta + se_{\tilde{\mathbf{k}}}) + (\delta + se_{\tilde{\mathbf{k}}})(d + si_{\mathbf{k}}) \\ &= d\delta + \delta d + s(\{\delta, i_{\mathbf{k}}\} + \{d, e_{\tilde{\mathbf{k}}}\}) + s^2\{i_{\mathbf{k}}, e_{\tilde{\mathbf{k}}}\} \,. \end{aligned}$$

The interior product of **k** with its dual is the function K^2, defined in Appendix 3(i), which only vanishes on the fixed submanifold. Calculation of the anti-commutators in the Hamiltonian may now be performed.

$$\{i_{\mathbf{k}}, e_{\tilde{\mathbf{k}}}\} = i_{\mathbf{k}}e_{\tilde{\mathbf{k}}} + e_{\tilde{\mathbf{k}}}i_{\mathbf{k}}$$

After using the Leibnitz Rule for an anti-derivation, this gives

$$\begin{aligned} \{i_{\mathbf{k}}, e_{\tilde{\mathbf{k}}}\} &= (i_{\mathbf{k}}\tilde{K}) - e_{\tilde{\mathbf{k}}}i_{\mathbf{k}} + e_{\tilde{\mathbf{k}}}i_{\mathbf{k}} \\ &= K^2 \,. \end{aligned}$$

Similarly

$$\begin{aligned} \{d, e_{\tilde{\mathbf{k}}}\} &= e_{d\tilde{\mathbf{k}}} - e_{\tilde{\mathbf{k}}}d + e_{\tilde{\mathbf{k}}}d \\ &= e_{d\tilde{\mathbf{k}}} \end{aligned}$$

and

$$\begin{aligned} \{\delta, e_{\tilde{\mathbf{k}}}\} &= -\{{}^*d^*, i_{\mathbf{k}}\} \\ &= -(-1)^p({}^*\{d, {}^*i^*_{\mathbf{k}}\}^*) \\ &= (-1)^p({}^*e^*_{d\tilde{\mathbf{k}}}) \end{aligned}$$

where ${}^*e^*_{d\tilde{\mathbf{k}}}$ acts as an interior product.

The full expression for the Hamiltonian is therefore,

$$H_s = d\delta + \delta d + s^2K^2 + s(e_{d\tilde{\mathbf{k}}} + (-1)^p({}^*e^*_{d\tilde{\mathbf{k}}}))$$

$e_{d\tilde{\mathbf{k}}}$ acts on a p-form to give a (p+2)-form and ${}^*e^*_{d\tilde{\mathbf{k}}}$ acts on a p-form to give a (p-2)-form.

3.1.1 Isolated Fixed Points

For s large the Hamiltonian is dominated by the potential term s^2K^2 which is large except near the zeroes of K^2, that is, the fixed point set of the Killing Vector. The low energy eigensolutions thus become concentrated around the fixed submanifold N. The case when N consists of isolated fixed points is similar to the case of non-degenerate Morse Theory discussed in sections 1 and 2, the Hamiltonian approximates to a harmonic oscillator around each zero of K^2. Using a locally Euclidean coordinate system and using the fermion creation and annihilation operator notation gives simple formulae for the relevant operators. The Killing Vector takes the form of a rotation around the fixed point. After rotating and translating the fixed point to the origin to simplify the formulae, and reflecting if necessary

$$k = \sum_{i=1}^{\frac{n}{2}} \lambda_i [x_{2i-1} \frac{\partial}{\partial x_{2i}} - x_{2i} \frac{\partial}{\partial x_{2i-1}}]$$

$$i_k = \sum_{i=1}^{\frac{n}{2}} \lambda_i [x_{2i-1} a_{2i} - x_{2i} a_{2i-1}]$$

where the x_i have been labelled such that **k** is in the standard form where $\lambda_i > 0$, for all i. The adjoint operator $e_{\tilde{\mathbf{k}}}$ takes the form

$$e_{\tilde{\mathbf{k}}} = \sum_{i=1}^{\frac{n}{2}} \lambda_i [x_{2i-1} a^*_{2i} - x_{2i} a^*_{2i-1}]$$

which leads to the formula for the exterior operator $e_{d\tilde{\mathbf{k}}}$

$$\begin{aligned} e_{\tilde{\mathbf{k}}} &= \sum_{j=1}^{n} \sum_{i=1}^{\frac{n}{2}} \lambda_i a^*_j \frac{\partial}{\partial x_j} [x_{2i-1} a^*_{2i} - x_{2i} a^*_{2i-1}] \\ &= \sum_{j=1}^{n} \sum_{i=1}^{\frac{n}{2}} \lambda_i [\delta_{j,2i-1} a^*_j a^*_{2i} - \delta_{j,2i} a^*_j a^*_{2i-1}] \\ &= \sum_{i=1}^{\frac{n}{2}} \lambda_i [a^*_{2i-1} a^*_{2i} - a^*_{2i} a^*_{2i-1}] \\ &= 2 \sum_{i=1}^{\frac{n}{2}} \lambda_i a^*_{2i-1} a^*_{2i} \,. \end{aligned}$$

The adjoint of this operator may be obtained by acting on the interior product operator i_k with the adjoint of the exterior derivative δ.

$$(-1)^p (^*e^*_{d\tilde{\mathbf{k}}}) = \sum_{j=1}^{n} \sum_{i=1}^{\frac{n}{2}} \lambda_i [-a^*_j \frac{\partial}{\partial x_j}][x_{2i-1} a_{2i} - x_{2i} a_{2i-1}]$$

$$= -2\sum_{i=1}^{\frac{n}{2}} \lambda_i a_{2i-1} a_{2i}$$

The function K² is

$$K^2 = \{i_k, e_{d\tilde{\mathbf{k}}}\} = \sum_{i,j=1}^{\frac{n}{2}} \lambda_i\lambda_j\{(x_{2i-1}a_{2i} - x_{2i}a_{2i-1}), (x_{2j-1}a^*_{2j} - x_{2j}a^*_{2j-1})\}$$

$$= \sum_{i,j=1}^{\frac{n}{2}} \lambda_i\lambda_j(\delta_{2i,2j}x_{2i-1}x_{2j-1} + \delta_{2i-1,2j-1}x_{2i}x_{2j})$$

$$= \sum_{i,j=1}^{\frac{n}{2}} \lambda_i^2(x^2_{2i-1} + x^2_{2i})$$

the anti-commutators $\{a_{2i}, a^*_{2j-1}\} = 0$, because 2i is even and 2j – 1 is odd.

Putting these various formulae together gives the approximate Hamiltonian, localized near an isolated fixed point at the origin as

$$\tilde{H}_s = -\sum_{i=1}^{\frac{n}{2}} \frac{\partial^2}{\partial x_i^2} + s^2\sum_{i=1}^{\frac{n}{2}} \lambda_i^2[x^2_{2i-1} + x^2_{2i}] + 2s\sum_{i=1}^{\frac{n}{2}} \lambda_i[a^*_{2i-1}a^*_{2i} - a_{2i-1}a_{2i}] \,.$$

This Hamiltonian has one zero energy eigensolution, which can be found as follows. Note that

$$d[\sum_{i=1}^{\frac{n}{2}} \lambda_i[x^2_{2i-1} + x^2_{2i}]] = \sum_{j=1}^{n}\sum_{i=1}^{\frac{n}{2}} \lambda_j a^*_j \frac{\partial}{\partial x_j}[x^2_{2i-1} - x^2_{2i}]$$

$$= 2\sum_{i=1}^{\frac{n}{2}} \lambda_i[a^*_{2i-1}x_{2i-1} + a^*_{2i}x_{2i}]$$

$$i_k[\sum_{i=1}^{\frac{n}{2}} a^*_{2i-1} + a^*_{2i}] = \sum_{i,j=1}^{\frac{n}{2}} \lambda_j[x_{2j-1}a_{2j} - x_{2j}a_{2j-1}]a^*_{2i-1}a^*_{2i}$$

$$= \sum_{i,j=1}^{\frac{n}{2}} \lambda_j[-\delta_{2j,2i}x_{2j-1}a^*_{2i-1} - \delta_{2j-1,2i-1}x_{2j}a^*_{2i}]$$

$$= -\sum_{i=1}^{\frac{n}{2}} \lambda_i[a^*_{2i-1}x_{2i-1} + a^*_{2i}x_{2i}]$$

Therefore this leads to a normalizable even-form state $\Psi_0(x)$ which is annihilated by $\tilde{d}_s$, the first order approximation to d_s .

$$\Psi_0(x) = exp[-\tfrac{s}{2}\sum_{i=1}^{\frac{n}{2}} \lambda_i[x^2_{2i-1} + x^2_{2i}]]exp[-\sum_{i=1}^{\frac{n}{2}} a^*_{2i-1}x^*_{2i}]|0>$$

where $|0>$ is the fermion Fock Space vacuum, the state annihilated by all the annihilation operators, and where the exponential in a^*_j is a finite series due to the finite dimension of the manifold,

$$exp[-\sum_{i=1}^{\frac{n}{2}} \lambda_i a^*_{2i-1} + a^*_{2i}] = 1 - \sum_{i=1}^{\frac{n}{2}} a^*_{2i-1}x^*_{2i} + \cdots + (-1)^{\frac{n}{2}+1}[\sum_{i=1}^{\frac{n}{2}} a^*_1a^*_2\cdots\bar{a}^*_{2i-1}\bar{a}^*_{2i}\cdots a^*_n] + (-1)^{\frac{n}{2}}[a^*_1a^*_2\cdots a^*_n]$$

where $\bar{a}^*_{2i-1}$ means that a^*_{2i-1} is omitted. Acting with the Hodge Star and using the locally flat metric

$$^*exp[-\sum_{i=1}^{\frac{n}{2}} a^*_{2i-1}a^*_{2i}] = [a^*_1 a^*_2 \cdots a^*_n] - \sum_{i=1}^{\frac{n}{2}} [a^*_1 \cdots \bar{a}^*_{2i-1}\bar{a}^*_{2i} \cdots a^*_n] + \cdots (-1)^{\frac{n}{2}+1} \sum_{i=1}^{\frac{n}{2}} a^*_{2i-1}a^*_{2i} + (-1)^{\frac{n}{2}}$$

$$= (-1)^{\theta}(-1)^{\frac{n}{2}} exp[-\sum_{i=1}^{\frac{n}{2}} a^*_{2i-1}a^*_{2i}]$$

where θ equals 0 or 1, depending on whether the orientation on the coordinate patch containing the fixed point agrees or disagrees with the natural orientation of the manifold. Therefore $\Psi_0(x)$ is self-dual or anti-self-dual, depending on the orientation of the patch and whether the dimension of the manifold is an even or an odd multiple of two. This implies that $\Psi_0(x)$ is also annihilated by δ_s to this approximation:

$$\tilde{\delta}_s \Psi_0(x) = -(^*\tilde{d}^*_s)\Psi_0(x) = \pm^*\tilde{d}_s \Psi_0(x) = 0$$

$\Psi_0(x)$ is the only normalizable state which is annihilated by $\tilde{d}_s$ and also by $\tilde{\delta}_s$, due to its self or anti-self-duality; it is therefore the only zero energy eigensolution of the approximate Hamiltonian H_s . All other eigensolutions have energies proportional to s.

Taking into account all the fixed points leads to the Lefschetz Fixed Point Theorem. The total number of even-form approximate zero energy solutions N_+, is equal to the number of fixed points. There are no odd-form zero energy solultions. Equating the value of the index of D_s, acting on the decomposed exterior algebra $\Lambda^{even}(M)$, $\Lambda^{odd}(M)$, when s = 0 and when s tends to infinity gives

$$\chi(M) = B_+(M) - B_-(M) = N_+ = \text{number of fixed points}$$

where $B_+(M)$, $B_-(M)$ are the sum of the even, odd Betti Numbers of M.

3.1.2 Non-isolated Fixed Points

These considerations generalize straightforwardly to the case where the fixed point set does not just consist of isolated fixed point. For large s the low energy solutions are localized around the fixed submanifold N. Near a connected component of the fixed submanifold N_0, coordinates may be chosen locally to give a product space of the (n-2q)-dimensional submanifold N_0, and 2q-dimensional Euclidean coordinates. In these coordinates the Hamiltonian takes the approximate form of the Laplacian on N_0 plus a 2q-dimensional harmonic oscillator of the same form as in the isolated fixed points case. The exterior derivative may be split into the exterior derivative on N_0, which anti-commutes with i_k, and a piece in the 2q flat coordinates which doesn't. The zero energy solutions $\Psi^i_0(x)$ in this approximation take the form of products of the representatives of the cohomology of N_0 with the zero energy solution of the 2q-dimensional harmonic oscillator

$$\Psi^i_0(x) = \chi^i(x) \exp[-\tfrac{s}{2}\sum_{j=1}^{q} \lambda_j [x^2_{2j-1} + x^2_{2j}]] \exp[-\sum_{j=1}^{q} [a^*_{2j-1}a^*_{2j}]|0>$$

where the $\chi^i(x)$ are the representatives of the cohomology of N_0. The fact that $\tilde{d}_s\Psi_0^i(x) = 0$ follows immediately from the fact that $\chi^i(x)$ is annihilated by the exterior derivative on N_0, and the harmonic oscillator ground state part of $\Psi_0^i(x)$ is annihilated by the rest of $\tilde{d}_s$. In this approximation the Hodge Star will act separately on each part of the product space, giving

$$^*\Psi_0^i(x) = (-1)^q[\tilde{*}\chi^i(x)]\exp[-\tfrac{s}{2}\sum_{j=1}^{q}\lambda_j[x_{2j-1}^2 + x_{2j}^2]]\exp[-\sum_{j=1}^{q}[a_{2j-1}^*a_{2j}^*]]|0>$$

where $\tilde{*}$ is the Hodge Star on N_0. $\tilde{*}\chi^i(x)$ must also be a representative of the cohomology of N_0 by Poincare duality; therefore $\tilde{d}_s\Psi_0^i(x) = 0$ implies $\tilde{\delta}_s\Psi_0^i(x) = 0$.

Taking into account all pieces of the fixed submanifold N, the total number of even-form approximate zero energy solutions N_+ is equal to the sum of the even Betti Numbers of all the components of N, and the total number of odd-form approximate zero energy solutions N_- is equal to the sum of the odd Betti Numbers of N. The fact that the index is independent of s gives,

$$\chi(M) = B_+(M) - B_-(M) = N_+ - N_- = \chi(N)$$

the Euler Characteristic of the whole manifold equals the Euler Characteristic of the fixed point set.

The results will be illustrated by analysis of supersymmetric quantum mechanics on the manifolds S^2 and $\mathbb{CP}^2$. These models will be analysed in detail, including examination of the excited states. In contrast to the zero energy states which are only dependent on the topology, the excited states depend on the geometry of the manifold. The zero energy states are singlets under the manifold's isometry group whereas the excited states form non-trivial representations. Distorting the manifold would alter the isometery group and so have a non-trivial effect on the excited state; similarly distorting the exterior derivative gives a Hamiltonian which is no longer invariant under the action of the full isometry group.

3.2 Bound on the Betti Numbers of the Fixed Point Set

Two other inequalities which follow from Witten's analysis are the bounds on the Betti Numbers of the fixed point set:

$$B_+(M) \geq N_+ \qquad , \qquad B_-(M) \geq N_- \, .$$

These inequalities may be demonstrated straightforwardly, as follows.

For any non-zero s the number of zero energy eigensolutions of H_s is independent of s and denoted $n_+ + n_-$. This quantity's independence of s is demonstrated by defining an operator $e^{\lambda P}$, which multiplies a p-form by $e^{\lambda p}$, and conjugating d_s by this operator:

$$\begin{aligned} e^{-\lambda P} d_s e^{\lambda P} &= e^{-\lambda P} d e^{\lambda P} + e^{-\lambda P} s i_k e^{\lambda P} \\ &= e^{-\lambda} d + e^{\lambda} s i_k \\ &= e^{-\lambda}(d + e^{2\lambda} s i_k) \\ &= e^{-\lambda}(d + s' i_k) \\ &= e^{-\lambda} d_{s'} \end{aligned}$$

where $s' = e^{2\lambda} s$. Conjugation by $e^{\lambda P}$ cannot change the dimension of the states closed but not exact in the sense of d_s , thus s may be changed to an arbitrary non-zero value without altering the number of zero energy eigensolutions of H_s .

As the continuous parameter s is varied the only value at which the number of zero energy eigensolutions may change is s = 0. The energy of an eigensolution expressed as a function of s, E(s), will vary continuously as s changes. If E(s) = 0 for all s > 0, it must be zero when s is equal to zero by continuity. However, if E(s) > 0 for all s > 0, it is possible that E(s) → 0 as s → 0, so it is possible for a supersymmetry quadruplet of states to have an energy which converges to zero as s vanishes. This implies that the number of zero energy eigensolutions when s = 0 must be greater than or equal to the number of zero energy eigensolutions for non-zero s, i.e.

$$B_+(M) + B_-(M) \geq n_+ + n_- \, .$$

Using the relation $n_+ + n_- = N_+ + N_-$ leads to bounds on the sum of the Betti Numbers of N.

$$B_+(M) + B_-(M) \geq N_+ + N_-$$

Along with the Lefschetz Fixed Point Theorem

$$\chi(M) = B_+(M) - B_-(M) = N_+ - N_- = \chi(N)$$

this gives the two inequalities:

$$B_+(M) \geq N_+ \qquad , \qquad B_-(M) \geq N_- \, .$$

Moreover the difference between these quantities must be even due to the fact that the solutions whose energy converges to zero must form supersymmetry quadruplets:

$$B_+(M) - N_+ = 2K \qquad , \qquad B_-(M) - N_- = 2K \, ,$$

where K is an integer. A case where these inequalities are not saturated is the torus T^2, see Appendix 3(iii).

3.3 The Hirzebruch Signature

Acting with the Hodge Star on the Schrödinger Equation

$$
\begin{aligned}
&-{}^*(d_s{}^*d_s^* + {}^*d_s^*d_s)\Psi(x) = E^*\Psi(x)\\
&-{}^*d_s^*d_s^*\Psi(x) + (-1)^p d_s^*d_s\Psi(x) = E^*\Psi(x)\\
&-({}^*d_s^*d_s + (-1)^p d_s^*d_s(-1)^{p*})^*\Psi(x) = E^*\Psi(x)\\
&-({}^*d_s^*d_s + d_s^*d_s{}^*)^*\Psi(x) = E^*\Psi(x)
\end{aligned}
$$

leads to the conclusion that if $\Psi(x)$ is a solution with energy E, then its dual ${}^*\Psi(x)$ is also solution with energy E. By taking linear combinations the solutions of the Schrödinger Equation may be taken to be self and anti-self-dual forms. All non-zero energy solutions must come in degenerate self and anti-self-dual pairs, so the number of self-dual solutions minus the number of anti-self-dual solutions must be independent of s for the same reason as in the previous cases. In fact, the only solutions which are not necessarily paired are the zero energy p-forms of middle degree.

When s = 0 the Hamiltonian is the ordinary Laplacian, so the number of self-dual solutions of the Schrödinger Equation minus the number of anti-self-dual solutions equals the trace of the quadratic form [9]

$$\sigma_{ij} = \int_M \Psi_i(x) \wedge \Psi_j(x)$$

where $\Psi_i(x), \Psi_j(x)$ are normalized representatives of the middle cohomology group of M. Decomposing the representatives of the middle cohomology group into self and anti-self-dual forms, $\Psi_i^{\pm}(x)$, diagonalises the quadratic form,

$$
\begin{aligned}
\sigma_{ij} = ({}^*\Psi_i^{\pm}(x), \Psi_j(x)) = \pm(\Psi_i^{\pm}(x), \Psi_j^{\pm}(x)) &= \pm 1 \quad \text{if } \; i = j\,,\\
&= 0 \qquad \text{if } i \neq j\,.
\end{aligned}
$$

where $(\Psi_i(x), \Psi_j(x))$ denotes the inner product of $\Psi_i(x)$ and $\Psi_j(x)$. The trace of the quadratic form is a topological invariant, the signature $\tau(M)$. If the forms of middle degree are odd-forms the quadratic form is anti-symmetric, so its trace is zero, therefore the signature is only non-zero for manifolds of dimension n = 4k. The signature may therefore be expressed in terms of the numbers of self and anti-self-dual harmonic 2k-forms:

$$\tau(M) = B_{2k}^+(M) - B_{2k}^-(M)$$

where the middle Betti Number has been decomposed,

$$B_{2k}(M) = B_{2k}^+(M) + B_{2k}^-(M)\,.$$

The number of self-dual minus anti-self-dual eigensolutions of the Hamiltonian must still be the same in the large s limit. In the case when all the fixed points are isolated, each one will be in a separate coordinate patch.

Taking a particular fixed point and labelling the coordinates x_i in the patch, such that the Killing Vector is in the standard form where all the λ_i are positive, the zero energy state localized near the fixed point will be self or anti-self-dual depending on whether $dx^i \wedge dx^2 \wedge \wedge dx^n$ is an even or odd multiple of the volume form at the fixed point. Assigning a sign n_i, equal to plus or minus one respectively, to each fixed point gives the result,

$$\tau(M) = \sum_{fixedpoints} n_i$$

When the fixed point set N doesn't consists of isolated points, an orientation form α_i may be assigned to an (n-2q)-dimensional component N_i, by requiring that the n-form $\alpha_i \wedge d\tilde{K} \wedge \cdots \wedge d\tilde{K}$, with q factors of $d\tilde{K}$, is a positive multiple of the volume form of M. In the large s limit, with the Hodge Star operation acting separately on the coordinates of N_i and the approximate flat transverse coordinates, states self or anti-self-dual under the Hodge Star on M correspond to eigensolutions of the Laplacian on N_i, self or anti-self-dual under the Hodge Star on N_i . Adding contributions to the index from each component of N gives:

$$\tau(M) = \tau(N)\,.$$

This will be illustrated in the case of supersymmetric quantum mechanics on $\mathbb{CP}^2$.

3.4 Relation to Quantum Field Theory

The supersymmetry operators may be slightly redefined as

$$Q_{s1} = i^{\frac{1}{2}} d_s + i^{-\frac{1}{2}} \delta_s \quad , \qquad Q_{s2} = i^{-\frac{1}{2}} d_s + i^{-\frac{1}{2}} \delta_s$$

which gives the supersymmetry algebra

$$\begin{aligned} Q_{s1}^2 &= d_s\delta_s + \delta_s d_s + i(d_s^2 - \delta_s^2) = H - P \\ Q_{s2}^2 &= d_s\delta_s + \delta_s d_s - i(d_s^2 - \delta_s^2) = H + P \\ \{Q_{s1}, Q_{s2}\} &= 0, [H, Q_{si}] = [P, Q_{si}] = [H, P] = 0 \end{aligned} \tag{1}$$

where $P = -2is\mathcal{L}_k$ is a central charge, in other words it commutes with all the other operators in the algebra. All the previous considerations about the zero energy states obviously still applies in this context as the Hamiltonian is unaltered.

Witten proves the Lefschetz Fixed Point Theorem in the finite dimensional, quantum mechanics case, but with a mind to ultimate application to quantum field theory which is infinite dimensional. These results are still on firm ground, because their proofs are only dependent on perturbation theory, which is the usual approach to the analysis of quantum field theories. In fact they only rely on the zeroth order perturbative result, equivalent to the Born Approximation, and so no complications due to regularization and renormalization arise.

In the case of the two-dimensional non-linear sigma model the supersymmetry operators obey the above algebra when the Killing Vector is taken to be the generator of translations in the spatial dimension, so that the central charge P is the momentum operator, and the algebra is the two-dimensional supersymmetry algebra. This σ-model may be treated as quantum mechanics on the infinite dimensional loop space $\Omega(B; S)$ of maps from the spatial dimension S into the target manifold B. The fixed point set of the translation is the target manifold B, so that if the Lefschetz Fixed Point Theorem is still valid, the index of the supersymmetry operator must equal the Euler Characteristic of the target manifold.

$$Tr(-1)^F = \chi(\Omega(B; S)) = \chi(B)$$

This implies that the non-linear sigma model on any manifold with non-zero Euler Characteristic has unbroken supersymmetry.

Witten's proof that, for $s \neq 0$ the number of zero eigensolutions of the Hamiltonian equals the sum of the Betti Numbers of the fixed point set, is not so obviously applicable in the quantum field theory case, as it relies on a non-perturbative construction. If this result does still hold it means that supersymmetry is never broken when the target manifold of the sigma model is compact and orientable, because at least two Betti Numbers must be non-zero.

Appendix 3(ii)

Odd-dimensional manifolds

In sections 3, 4 and 5 only the even-dimensional manifolds will be considered as the Lefschetz Fixed Point Theorem is trivial in the odd-dimensional case. Each harmonic p-form $\chi_p(x)$ is related by Poincare duality to a harmonic (n-p)-form ${}^*\chi_p(x)$:

$$(d\delta + \delta d)\chi_p(x) = 0 \Longrightarrow \pm^*(d^*d^* + {}^*d^*d)^{**}\chi_p(x) = 0$$
$$\pm({}^*d^*d + d^*d^*)^*\chi_p(x) = 0$$

so the Betti Numbers of a manifold obey the relationship

$$B_p(M) = B_{n-p}(M), \qquad 0 \leq p \leq n \, .$$

On an odd-dimensional manifold Poincare duality relates harmonic even-forms to harmonic odd-forms, so the Euler Characteristic is always zero. The dimension of the fixed point set N always equals the dimension of M minus an even number, so the fixed point set of an odd-dimensional manifold is odd-dimensional. Therefore, in the odd-dimensional case, the Lefschetz Fixed Point Theorem relates two quantities which must be zero anyway.

Appendix 3(iii)

A simple example: T²

T² has two commuting continuous isometries generated by the Killing Vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Introducing the Killing Vector $\frac{\partial}{\partial x}$ into the supersymmetry algebra

$$d_s = d + s i_{\frac{\partial}{\partial x}} = \begin{pmatrix} 0 & s & 0 & 0 \\ \frac{\partial}{\partial x} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & 0 & 0 & s \\ 0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}$$

$$\delta_s = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 \\ s & 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & 0 & s & 0 \end{pmatrix}$$

$$H_s = d_s\delta_s + \delta_s d_s = [-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + s^2]\mathbb{I}$$

where $\mathbb{I}$ is the 4x4 unit matrix.

The eigensolutions of the Schrödinger Equation are still the same as in the cse without the introduction of the Killing Vector, but the energy levels are raised. The spectrum is now

$$E = \frac{4\pi^2}{a^2}(n^2 + m^2) + s^2$$

where n and m are integers. The ground state energy is

$$E_0 = s^2$$

so there are no zero energy solutions. This is due to the fact that the Killing Vector is a translation in $\mathbb{R}^2$ and has no fixed points on the torus.

This is an example where the inequalities,

$$B_+(M) \geq N_+ \qquad , \qquad B_-(M) \geq N_- \, ,$$

the bounds on the Betti Numbers of the fixed point set, are not saturated. In fact

$$B_+(T^2) = 2 \qquad , \qquad N_+ = 0$$

$$B_-(T^2) = 2 \qquad , \qquad N_- = 0$$

the four eigensolutions which are harmonic in the sense of the ordinary exterior derivative form a supersymmetry quadruplet when $d \to d_s$.

Because there are no zero energy solutions supersymmetry is broken. In quantum field theories, in this case, the non-renormalization theorem would no longer hold, so fermion and boson masses would no longer be cancel. However, in quantum mechanics, as there is no such renormalization, energy levels will still be boson-fermion degenerate even if there are no zero energy solutions.