Supersymmetric Quantum Mechanics and Geometry

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October 1989

Preface

This dissertation is the result of my own individual effort, except where reference is explicitly made to the work of others, and includes nothing which is the outcome of work done in collaboration. It is not the same as any work that has been submitted at any other university.

The work was carried out at the Department of Applied Maths and Theoretical Physics under the supervision of Dr N. S. Manton.

I would like to greatly thank Dr N. S. Manton for all his assistance.

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Chapter 1

Supersymmetry and Morse Theory

1. Supersymmetry and Morse Theory

Various of Witten's papers of the early 1980s [1], [2], [3] were concerned with the subject of supersymmetry breaking. The fermion-boson mass degeneracy produced by supersymmetry is not observed in nature, so in any supersymmetric theory, which purports to be a viable physical theory, supersymmetry must be broken. It is therefore important to know in which theories supersymmetry breaking occurs. The novelty of Witten's approach to this problem was to use the relationship between the indes of the supersymmetry operator and deformation invariants of the underlying manifold. The reasoning that pointed in this direction is as follows.

In quantum field theories spontaneous symmetry breaking occurs when the vacuum is not annihilated by the generators of the symmetry. The Hamiltonian in supersymmetric theories is equal to the sum of the squares of the supersymmetry generators and so spontaneous supersymmetry breaking occurs if and only if the vacuum energy is greater than zero. In supersymmetric theories non-renormalization theorems guarantee the cancellation of contributions to the vacuum energy from fermion and boson loop diagrams, so if supersymmetry is unbroken at the tree level the question of whether supersymmetry really is broken or not becomes a question about non-perturbative effects. The fact that perturbation theory is only concerned with local information, whereas non-perturbative effects incorporate global information points to the conclusion that spontaneous supersymmetry breaking is related to the topology of the manifold on which the supersymmetry operators are defined. The index of the supersymmetry operator is a topological invariant and equals $Tr(-1)^F$ the number of zero energy bosons minus the number of zero energy fermions. If the index is non-zero there must be a zero energy state, so supersymmetry is unbroken, if the index is zero there may or may not be zero energy states.

In general the states annihilated by a particular differential operator are not easily found. One approach to finding whether such states exist would be to distort the manifold continuously into an orbifold, so that now all the curvature and therefore all the topological information is located at isolated points. The index of the elliptical operator will be equal to a topological invariant of the manifold, and this topological invariant will be the same for the orbifold as for the manifold, but in the case of the orbifold it will be calculable with reference only to the isolated points with non-zero curvature.

In Witten's paper 'Supersymmetry and Morse Theory' an alternative approach is taken. Instead of distorting the base manifold, the elliptic operator defined on it is distorted. In the first case, the supersymmetry operator is conjugated by a Morse Function multiplied by an arbitrary parameter and the limit is taken as the parameter tends to infinity. The zero energy solutions of the Hamiltonian now become localized around isolated points (in the non-degenerate case), the critical points of the Morse Function, and are easily calculable. The fact that the index of the supersymmetry operator is independent of the arbitrary parameter leads to a proof of the Morse Inequalities, using "physicists' methods". After analysing quantum mechanical tunnelling between critical points, which removes some of the zero energy degeneracy, Witten proposes a strengthening of the Morse Inequalities.

In the second part of Witten's paper the supersymmetry operator is altered by the introduction of a Killing Vector multiplied by an arbitrary parameter. When the limit is taken of the parameter tending to infinity, the zero energy solutions of the Hamiltonian become localized around the fixed point set of the Killing Vector, where the supersymmetry operator still takes its unaltered form. The index's independence of the parameter leads to proofs of the Lefschetz Fixed Point Theorems, that the Euler Characteristic and Signature of the fixed point set equal those of the whole manifold. Witten also proves that the number of zero energy solutions for arbitrary non-zero values of the parameter is equal to the number of zero energy solutions for asymptotically large values of the parameter, and is thus equal to the sum of the Betti numbers of the fixed point set. Even though these results are proved in terms of finite dimensional quantum mechanics, they are ultimately intended for application to quantum field theory, which is infinite dimensional.

In what follows not much mention will actually be made of spontaneous supersymmetry breaking, but Witten's paper 'Supersymmetry and Morse Theory' will be examined in detail and various aspects of it will be illustrated by studying particular quantum mechanical examples. In the first section, the part of Witten's paper related to proving the Morse Inequalities, will be summarised. The second section is an illustration of these results using specific examples of Morse Functions on the two-dimensional torus, one of which saturates the Morse Identities and one which does not. The third section summarises the part of Witten's paper relating to Killing Vectors and their fixed point sets. In the fourth section the results about Fixed Point Theorems will be illustrated in the case where the fixed point set consists of isolated points. This will be done by analysing the zero energy states of supersymmetric quantum mechanics on the two-dimensional sphere. The effect on the excited states of introducing a Killing Vector into the supersymmetry algebra will also be studied using various approximate methods. The fifth section is an illustration of the Fixed Point Theorems when the fixed point set does not necessarily consist of isolated points. The case which will be treated is that of supersymmetry on the manifold $\mathbb{CP}^2$. The effect of introducing a Killing Vector will again be studied in detail. Various standard results and definitions pertaining to topology and complex manifolds will appear in the appendices.

1.1 Supersymmetric Quantum Mechanics

The simplest possible supersymmetry algebra is the N = 2 supersymmetric quantum mechanics algebra:

$$H = Q_1^2 = Q_1^2 \qquad , \qquad \{Q_1, Q_2\} = 0$$

There are two supersymmetry operators due to the fact that the quantum mechanical wavefunction is always complex.

One way to construct operators which satisfy this algebra is in terms of the De Rham operators, acting on the exterior algebra Λ*(M), of the manifold, thus leading to connections between supersymmetric quantum mechanics and the cohomology of M. Proceeding in this manner gives the supersymmetry operators as:

$$Q_1 = d + \delta \quad , \qquad Q_2 = i(d - \delta)$$

Where d is the exterior derivative and δ is its adjoint, see Appendix 1a, $\delta = (-1)^{np+n+1*}d^*$, where n is the dimension of the manifold and p is the degree of the form on which δ is acting, * is the Hodge Star operation. Squaring these operators gives the Hamiltonian as equal to the Laplacian:

$$H = \triangle = d\delta + \delta d$$

The supersymmetry operators map even-forms to odd-forms and odd-forms to even-forms, splitting the exterior algebra into $\Lambda^{even}(M)$ and $\Lambda^{odd}(M)$. Even-forms may be thought of as bosons and odd-forms as fermions. Non-zero energy solutions of the Schrödinger Equation are generically doubly degenerate. Using the Hodge Decomposition, a wavefunction may be split into an exact and a coexact piece

$$\Psi_p = d\alpha_{p-1} + \delta\beta_{p+1}$$

the harmonic piece must vanish for non-zero energy.

Substituting this wavefunction into the Schrödinger Equation gives

$$(d\delta + \delta d)(d\alpha_{p-1} + \delta\beta_{p+1}) = E(d\alpha_{p-1} + \delta\beta_{p+1})$$

$$d(\delta d\alpha_{p-1}) + \delta(d\delta\beta_{p+1}) = E(d\alpha_{p-1} + \delta\beta_{p+1}) \quad .$$

Equating the exact and coexact pieces

$$d\delta d\alpha_{p-1} = E d\alpha_{p-1} \ , \qquad \delta d\delta\beta_{p+1} = E\delta\beta_{p+1}$$

leads to the conclusion that $d\alpha_{p-1}$ and $\delta\beta_{p+1}$ are separately eigensolutions of the Hamiltonian with energy E. We can now assume that Ψ is either exact or coexact. If it is exact

$$\Psi_p = d\alpha_{p-1}$$

acting with Q_1 gives

$$Q_1\Psi_p = \delta d\alpha_{p-1}$$

and acting with Q_2 gives the same result, so supersymmetry pairs an exact p-form to a coexact (p-1)-form. These two solutions are degenerate due to the commutivity of the supersymmetry operators with the Hamiltonian. Alternativaly, if the wavefunction was a coexact p-form it would be paired by supersymmetry to an exact (p+1)-form.

Zero energy solutions of the Schrödinger Equation are annihilated by the supersymmetry operators and so do not form doublets. These solutions correspond to the harmonic p-forms, and the number of independent such p-form solutions is equal to the Betti Number $B_p(M)$, the dimension of the p-th cohomology group of the manifold.

The Witten Index $Tr(-1)^F$, the difference in the number of bosons and fermions in the theory, in fact equals the number of zero energy bosons minus the number of zero energy fermions, as all non-zero energy solutions come in boson-fermion pairs. In the case of supersymmetric quantum mechanics the Witten Index is the index of the De Rham complex, and thus equals the Euler Characteristic of the base manifold.

$$Tr(-1)^F = index(\Lambda^{even,odd}, d+\delta) = \sum_{p=0}^{n}(-1)^p B_p = \chi(M) \ .$$

1.2 Morse Theory

Morse Theory relates the properties of the critical point set of a Morse Function, defined on a differentiable, compact Riemannian manifold M, to the underlying topology of the manifold. A Morse Function $h(x)$ is a smooth, real valued function on M

$$h(x) : M \longrightarrow \mathbb{R}$$

A critical point $\mathbf{x}_{\mathbf{crit}}$, is a point at which all the first derivatives of the Morse Function simultaneously vanish. The Morse Index p, of the critical point, is the number of negative eigenvalues of the matrix of second derivatives of the Morse Function at the critical point, known as the Hessian. Non-degenerate Morse Theory is concerned with Morse Functions which have only isolated critical points. For this to be the case the determinant of the Hessian must be non-zero for each critical point. A degenerate Morse Function has a Hessian which has zero eigenvalues for some of its extrema, the dimension of the critical manifold equals the number of zero eigenvalues. Unless specifically stated, what follows will only deal with non-degenerate Morse Theory.

1.2.1 The Weak Morse Inequalities

Witten's technique of incorporating a Morse Function into the supersymmetric quantum mechanics algebra is through conjugation of the exterior derivative and its adjoint

$$d_t = e^{-th(x)} d e^{th(x)} \qquad , \qquad \delta_t = e^{th(x)} \delta e^{-th(x)}$$

where t is an arbitrary parameter. The supersymmetry operators are now defined in terms of d_t and δ_t

$$Q_{1t} = d_t + \delta_t \ , \qquad Q_{2t} = i(d_t - \delta_t)$$

and the Hamiltonian is now

$$H_t = d_t \delta_t + \delta_t d_t \qquad .$$

The supersymmetry algebra is unchanged

$$Q_{1t}^2 = Q_{2t}^2 = H_t \qquad , \qquad \{Q_{1t}, Q_{2t}\} = 0$$

As $t \longrightarrow 0$ the Hamiltonian H_t tends to the ordinary Laplacian, so for t = 0 the number of zero energy p-form eigensolutions $\Psi_p(x)$ is equal to the Betti Number $B_p(M)$. For $t \neq 0$ each harmonic p-form $\Psi_p(x)$ is in a one-to-one correspondence with a p-form $e^{-th(x)}\Psi_p(x)$ which is closed, but not exact, in the sense of d_t . The numbers of zero energy p-form eigensolutions of H_t for $t \neq 0$ are therefore still equal to the Betti Numbers of the manifold.

In calculating an explicit formula for the Hamiltonian it is convenient to represent the exterior derivative as $a^{i*}\frac{D}{Dx^i}$, where the a^{i*} act on the exterior algebra as exterior products and the covariant derivative D/Dx^i acts on functions. The index i is summed over. The interior derivative is represented as $-a^j\frac{D}{Dx^j}$, the a^j being the adjoints of a^{j*}, acting on the exterior algebra as interior products. The a^{i*} and a^j may be taken to be fermion creation and annihilation operators as they change fermion number by plus and minus one respectively, and they satisfy the anti-commutation relations

$$\{a^{i*}, a^j\} = \gamma^{ij}$$

γ^{ij} is the metric tensor of the manifold. In terms of these operators the Hamiltonian takes the form

$$H_t = e^{-th}de^{2th}\delta e^{-th} + e^{th}\delta e^{-th}de^{th}$$

$$= d\delta + \delta d - (e^{-th}a^{i*}\frac{D}{Dx^i}e^{2th}a^j\frac{D}{Dx^j}e^{-th}) - (e^{th}a^j\frac{D}{Dx^j}e^{-2th}a^{i*}\frac{D}{Dx^i}e^{th})$$

+ terms with one derivative acting on the wavefunction.

The brackets mean that the derivatives act only on terms within the bracket and not on the wavefunction. The terms with one derivative acting on the wavefunction cancel thus

$$(e^{-th}a^{i*}\frac{D}{Dx^i}e^{th})a^j\frac{D}{Dx^j} + (e^{th}a^{i*}a^j\frac{D}{Dx^j}e^{-th})\frac{D}{Dx^i} + (e^{th}a^j\frac{D}{Dx^j}e^{-th})a^{i*}\frac{D}{Dx^i} + (e^{-th}a^ja^{i*}\frac{D}{Dx^i}e^{th})\frac{D}{Dx^j}$$

$$= ta^{i*}a^j\frac{D}{Dx^i}\frac{D}{Dx^j} - ta^{i*}a^j\frac{D}{Dx^j}\frac{D}{Dx^i} - ta^ja^{i*}\frac{D}{Dx^j}\frac{D}{Dx^i} + ta^ja^{i*}\frac{D}{Dx^i}\frac{D}{Dx^j}$$

$$= t\{a^{i*}, a^j\}\frac{\partial}{\partial x^i}\frac{D}{Dx^j} - t\{a^{i*}, a^j\}\frac{\partial}{\partial x^j}\frac{D}{Dx^i}$$

$$= 0$$

Leaving the Hamiltonian as

$$H_t = d\delta + \delta d + t^2 a^{i*} a^j \frac{Dh}{Dx^i}\frac{Dh}{Dx^j} + t^2 a^j a^{i*} \frac{Dh}{Dx^j}\frac{Dh}{Dx^i} + t a^{i*} a^j \frac{D^2 h}{Dx^i Dx^j} - t a^j a^{i*} \frac{D^2 h}{Dx^i Dx^j}$$

$$= d\delta + \delta d + t^2 \{a^{i*}, a^j\} \frac{Dh}{Dx^i}\frac{Dh}{Dx^j} + t[a^{i*}, a^j] \frac{D^2 h}{Dx^i Dx^j}$$

$$= d\delta + \delta d + t^2 \gamma^{ij} \frac{\partial h}{\partial x^i}\frac{\partial h}{\partial x^j} + t \frac{D^2 h}{Dx^i Dx^j}[a^{i*}, a^j]$$

For large t the potential term $t^2 \gamma^{ij} \frac{\partial h}{\partial x^i}\frac{\partial h}{\partial x^j}$ dominates the Hamiltonian, and the low energy eigensolutions become concentrated in the regions where this term is small, that is, near the critical point set where $\frac{\partial h}{\partial x^i} = 0$ for all x^i . In the case of non-degenerate Morse Theory, where the critical points are isolated, the Hamiltonian tends to the form of a supersymmetric harmonic oscillator around each critical point. Taking locally Euclidean coordinates and approximating the Morse Function near a critical point as

$$\tilde{h}(x_i - x_{crit}) = h(x_{crit}) + \tfrac{1}{2}\lambda_i (x_i - x_{crit})^2$$

with summation over I, where λ_i and x_i are the eigenvalues and eigenvectors respectively, of the Hessian. The Hamiltonian takes the approximate form near the critical point

$$\tilde{H}_t = -\tfrac{\partial^2}{\partial x_i^2} + t^2 \frac{\partial \tilde{h}}{\partial x^i}\frac{\partial \tilde{h}}{\partial x^j} + t \frac{\partial^2 \tilde{h}}{\partial x^i \partial x^j}[a^{i*}, a^j]$$

where in the Euclidean approximation $a^{i*} = dx^i$ and $a^j = \frac{\partial}{\partial x^j}$.

$$\tilde{H}_t = -\tfrac{\partial^2}{\partial x_i^2} + t^2 \lambda_i^2 (x_i - x_{crit})^2 + t\lambda_i [a^{i*}, a^j]$$

The approximate Schrödinger Equation near the critical point has one zero energy solution, a p-form if there are p negative eigenvalues λ_i, which means the critical point has Morse Index p. All other solutions have energies proportional to t. The simplest way to demonstrate the existence of the zero energy solution is through its annihilation by the approximated supersymmetry operator.

Near a critical point of the Morse Index p the conjugated exterior derivative takes the approximate form

$$\tilde{d}_t = d + t\lambda_i (x_i - x_{crit}) dx_i$$

with summation over i. $\tilde{d}_t$ annihilates the state

$$\tilde{\Psi}_p(x) = exp[-\tfrac{1}{2}|\lambda_i|(x_i - x_{crit})^2] dx_j \wedge \cdots \wedge dx_k$$

which is formed from the harmonic oscillator ground state and a p-form composed of the exterior product of the one-form dx_j, taking only the coordinates x_j for which the corresponding eigenvalue λ_j is negative.

$$\tilde{d}_t \tilde{\Psi}_p(x) = [-t|\lambda_q|(x_q - x_{crit})dx_q + t\lambda_q (x_q - x_{crit})dx_q] exp[-\tfrac{t}{2}|\lambda_i|(x_i - x_{crit})^2] dx_j \wedge \cdots \wedge dx_k$$

The dx_q only run over the coordinates where λ_q is positive, as the other terms are annihilated when taking the exterior product, so $\tilde{d}_t \tilde{\Psi}_p(x) = 0$. Near the critical point the interior derivative takes the approximate form

$$\tilde{\delta}_t = \delta - t\lambda_i(x_i - x_{crit})\frac{\partial}{\partial x_i}$$

$\frac{\partial}{\partial x_i}$ acting on the exterior algebra. Acting on $\tilde{\Psi}_p(x)$ gives

$$\tilde{\delta}_t \tilde{\Psi}_p(x) = [-t|\lambda_k|(x_k - x_{crit})\frac{\partial}{\partial x_k} + t\lambda_k(x_k - x_{crit})\frac{\partial}{\partial x_k}]exp[-\tfrac{t}{2}|\lambda_i|(x_i - x_{crit})^2]dx_j \wedge \cdots \wedge dx_k$$

This time the dx_k only run over the coordinates for which λ_k is negative, as all the other terms are annihilated when taking the inner product, therefore $\tilde{\Psi}_p(x)$ is annihilated by $\tilde{\delta}_t$ as well as $\tilde{d}_t$. $\tilde{\Psi}_p(x)$ is therefore a zero energy eigensolution of the approximate Hamiltonian localized around the critical point, moreover it is the only zero energy eigensolution as no other exterior differential form is annihilated by both $\tilde{d}_t$ and $\tilde{\delta}_t$.

There is one approximate zero energy p-form solution concentrated around each critical point of Morse Index p. Taking into account all the critical points of the Morse Function the total number of approximate zero energy p-form solutions is equal to the Morse Number M_p, the number of critical points of Morse Index p. Each zero energy eigensolution of the exact Hamiltonian will approximate, in the large t limit, to a zero energy eigensolution of the approximate Hamiltonian, so the total number of approximate zero energy eigensolutions must be at least as large as the number of exact zero energy solutions. This gives immediately the weak form of the Morse Inequalities.

$$M_p \geq B_p(M) \qquad , \qquad 0 \leq p \leq n$$

The generalization of the preceding to the case of degenerate Morse Functions is straightforward. Near a critical manifold, in the large t limit, the Hamiltonian reduces to the ordinary Laplacian on the critical manifold, plus a harmonic oscillator in the transverse directions. The zero energy solutions of the approximate Schrödinger Equation around a critical manifold of Morse Index p are therefore the products of the p-form of the previous analysis and representatives of the cohomology of the critical manifold. The Morse Numbers may be defined more generally as

$$M_p = \sum_{N_k} B_{p-k}(N_k)$$

where the summation is over the critical manifolds N_k with Morse Index k. With this generalized definition all the previous results follow in a completely analogous way for degenerate Morse Functions.

1.2.1 The Strong Morse Inequalities

All non-zero energy solutions come in boson-fermion pairs. In taking the large t limit the energy of certain solutions may converge to zero, but any solutions whose energies, as a function of t, exhibit such behaviour must come in supersymmetry doublets, so the index of the supersymmetry operator $Q_{1t} = d_t + \delta_t$, acting on the decomposed exterior algebra $\Lambda^{even}_{(M)}, \Lambda^{odd}_{(M)}$, must be independent of t. When $t = 0$, the index equals the Euler Characteristic of the manifold. After taking the large t limit the number of zero energy solutions may be altered, but the index must be unchanged. From this it follows that

$$\sum_{p=0}^{n}(-1)^p M_p = \sum_{p=0}^{n}(-1)^p B_p(M) = \chi(M)$$

where n is the dimension of the manifold. Along with the weak form of the Morse Inequalities, this equation implies the strong form. Writing out the previous equation more explicitly

$$M_n - M_{n-1} + M_{n-2} - \cdots \pm M_0 = B_n - B_{n-1} + B_{n-2} - \cdots \pm B_0$$

and using the first of the Weak Morse Inequalities, $M_N \geq B_n$, gives

$$-M_{n-1} + M_{n-2} - \cdots \pm M_0 \leq -B_{n-1} + B_{n-2} - \cdots \pm B_0$$

$$M_{n-1} - M_{n-2} + \cdots \mp M_0 \geq B_{n-1} - B_{n-2} + \cdots \mp B_0 \, .$$

Each of the $(M_n - B_n)$ approximate zero energy n-form solutions which do not correspond to an exact zero energy solution must be paired by the action of δ_t with one of the $(M_{n-1} - B_{n-1})$ (n-1)-form approximate zero energy solutions which do not correspond to exact ones. However, some of these $(M_{n-1} - B_{n-1})$ solutions may be paired with (n-2)-form solutions, therefore

$$M_n - B_n \leq M_{n-1} - B_{n-1}$$

Substituting this inequality into the index formula gives

$$M_{n-2} - M_{n-3} + \cdots \pm M_0 \geq B_{n-2} - B_{n-3} + \cdots \pm B_0$$

Reasoning in the same manner implies

$$(M_n - B_n) + (M_{n-2} - B_{n-2}) \geq M_{n-1} - B_{n-1}$$

as some of the $(M_{n-2} - B_{n-2})$ approximate zero energy solutions which do not correspond to exact ones will be paired with (n-3)-form solutions. When this inequality is substituted into the index formula, the result is

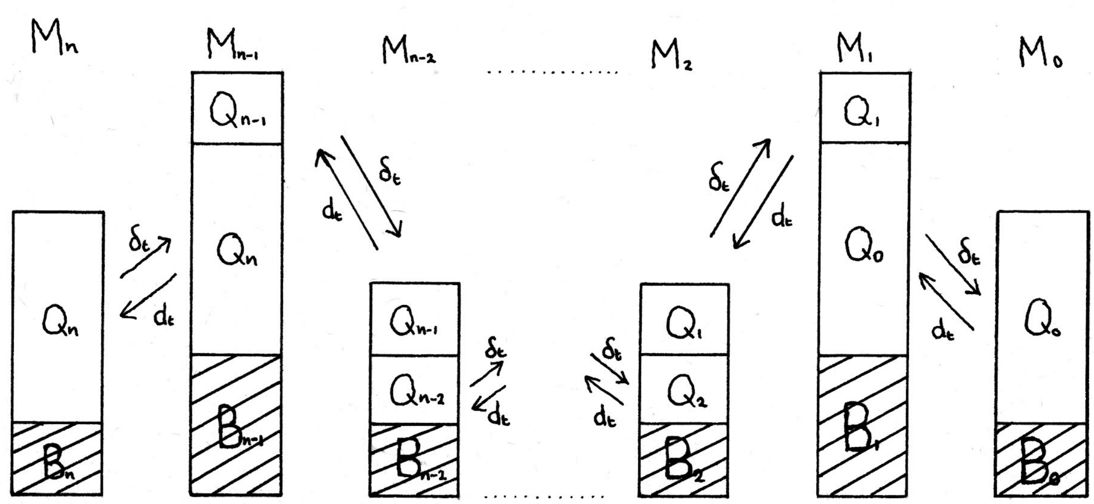
$$M_{n-3} - M_{n-4} + \cdots \mp M_0 \geq B_{n-3} - B_{n-4} + \cdots \mp B_0$$

Continuing in the same way leads to the whole of the strong form of the Morse Inequalities

$$\sum_{p=0}^{m} (-1)^{m-p} M_p \geq \sum_{p=0}^{m} (-1)^{m-p} B_p(M) \qquad , \qquad 0 \leq m \leq n$$

with equality when $m = n$.

A diagrammatical way of illustrating this result is as follows:



The columns represent the Morse Numbers M_p, the dimensions of the spaces of approximate zero energy p-form solutions. The shaded portions represent the Betti Numbers $B_p(M)$, the dimensions of spaces of solutions which are actually annihilated by the exact supersymmetry operator. The unshaded portions marked with the Q_ps represent the dimension of the spaces of approximate zero energy solutions which do not correspond to exact zero, and the arrows show how these spaces are mapped into each other by the action of the supersymmetry operator.

The strong form of the Morse Inequalities may be read from the diagram immediately. In the sum $\sum_{p=0}^{m} (-1)^{m-p} M_p$, all the Qs cancel except Q_p, so that

$$\sum_{p=0}^{m}(-1)^{m-p}M_p = \sum_{p=0}^{m}(-1)^{m-p}B_p(M) + Q_p \qquad .$$

The statement of the Morse Inequalities given by Witten, that

$$\sum_{p=0}^{n} M_p t^p - \sum_{p=0}^{n} B_p(M)t^p = (1+t)\sum_{p=0}^{n} Q_p t^p$$

is completely equivalent. Matching powers of t, this equation gives the following

$$\begin{aligned} &M_0 - B_0(M) = Q_0 \\ &M_p t^p - B_p(M)t^p = (Q_{p-1} + Q_p)t^p \quad , \qquad 1 \le p \le n-1 \\ &M_n t^n - B_n(M)t^n = Q_n t^n \end{aligned}$$

which, after cancelling the powers of t, are exactly the equations represented by the diagram.

1.2.3 Perturbation Theory

The spurious degeneracy of the zero energy solution is not removed in any order of perturbation theory. All the approximate zero energy solutions remain with zero energy, because perturbation theory only uses local information and so only involves a single critical point. As the terms in the Hamiltonian are expanded near a critical point, in powers of $\frac{1}{t}$, there always exists one zero energy eigensolution

$$\tilde{\Psi}_p(x) = exp[-t\tilde{h}(x_i - x_{crit})]dx_j \wedge \cdots \wedge dx_k$$

where $\tilde{h}(x_i - x_{crit})$ is the Morse Function expanded around the critical point to the appropriate order in $\frac{1}{t}$. This state will always be annihilated by both

$$\tilde{d}_t = d + t\frac{\partial \tilde{h}(x_i - x_{crit})}{\partial x_j}dx_j$$

and its adjoint

$$\tilde{\delta}_t = d - t\frac{\partial \tilde{h}(x_i - x_{crit})}{\partial x_j}\frac{\partial}{\partial x_j}$$

1.2.4 The W.K.B. Approximation

In order to improve the bound on the number of zero energy solutions a calculation must be performed which is sensitive to the existence of more than one critical point; that is, a calculation to determine the amplitude for tunnelling between critical points. In a more accurate calculation some of the Q_p previously zero energy p-form solutions which do not correspond to exact zero energy solutions, may be mapped to some of the Q_{p+1} previously zero energy (p+1)-form solutions which do not correspond to exact zero energy solutions, by the operator d_t, thus removing some of the zero energy degeneracy.

In the Feynman Path Integral approach to quantum mechanics all the possible paths between critical points contribute to the tunnelling amplitude, weighted by the exponential of minus the action evaluated for the particular path. The first approximation to this tunnelling amplitude is the semi-classical W.K.B. approximation. In this approximation the classical limit, Planck's constant $\hbar \rightarrow 0$ is taken, this is equivalent to the limit $t \rightarrow \infty$ in our case, so that only the contribution of the classical trajectory, which minimises the action, or path of steepest descent between critical points, is taken into account.

Taking approximate zero energy states $|a>$ and $|b>$ localized near critical points A and B of Morse Index p and p+1 respectively. $|a>$ will fall off like $exp[-th(x)]$ in ascending along the path of steepest descent Γ, from A to B, so that along this path the overlap between the states will be

$$< b|d_t|a >_{\Gamma}= exp(-t(h(B) - h(A)))$$

Summing over all the paths of steepest descent between A and B, gives

$$\begin{aligned} < b|d_t|a > &= \sum_{\Gamma_i} (-1)^{n_{\Gamma_i}} e^{-t(h(B)-h(A))} \\ &= n(A, B) e^{-t(h(B)-h(A))} \end{aligned}$$

where n_{Γ_i} equals 1 or 0 depending on the orientation of the path Γ_i.

Taking into account all critical points of Morse Index p+1, the action of d_t on the p-form state $|a>$, in this approximation is

$$\hat{d}|a >_p= \sum_b e^{-t(h(B)-h(A))} n(A, B)|b >_{p+1}$$

The states for which this expression cancels will remain zero energy singlets, while some previous approximately zero energy states may now form supersymmetry doublets under the action of this operator. As far as cohomology is concerned the factors of $e^{-t(h(B)-h(A))}$, which are the same along any path connecting A and B, may be dropped in the definition of the coboundary operator. Witten defines the coboundary operator $\triangle$, where

$$\triangle|a >_p= \sum_b n(A, B)|b >_{p+1}$$

whose action is completely determined by the orientation of paths of steepest descents between critical points. The adjoint of $\triangle$, the operator $\triangle^\star$, relates to δ_t in the W.K.B. approximation in a completely analogous way. Its definition is

$$\triangle^\star|a>_p=\sum_c n(A,C)|c>_{p-1}$$

mapping p-forms to (p-1)-forms. For a state to still have zero energy in this approximation it must be annihilated both by the coboundary operator $\triangle$ and its adjoint.

Calling the number of zero energy p-form states in the W.K.B. approximation Y_p; the arguments relating to the Morse Numbers M_p, and the Morse Inequalities follow for these numbers in exactly the same way. They must form upper bounds on the Betti Numbers of M

$$M_p \geq Y_p \geq B_p(M) \quad , \qquad 0 \leq p \leq n$$

and must satisfy an analogous formula to the Strong Morse Inequalities

$$\sum_p Y_p t^p - \sum_p B_p(M) t^p = (1+t) \sum_p Q_{p'} t^p \, , \qquad 0 \leq p \leq n$$

where $Q_{p'} \leq Q_p$.

Witten conjectures that the Y_p are in fact always equal to the Betti Numbers $B_p(M)$. In terms of the quantum mechanics view point this would seem a reasonable result, as degeneries tend to be removed by a first order tunnelling calculation. From the point of view of topology, however, this would be a remarkable refinement of Morse Theory. The critical points of a Morse Function give approximate information about the cohomology of a manifold; Witten aims through the properties of the critical points, plus the relative orientation of paths between critical points to gain exact knowledge of the cohomology of the manifold.

This section will be illustrated by examples of Morse Functions on the 2-torus, T^2.